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Ginzburg-Landau vortices, by F. Bethuel, H. Brezis, and F. Helein, Progress in Nonlinear Differential Equations & Their Appl., vol. 13, Birkhäuser, Basel and Boston, 1994, xxvii + 158 pp., \$49.50, ISBN 0-8176-3723-0

This book is concerned with singular limits, as $\epsilon \to 0$, of solutions $u(x;\epsilon)$ of the Ginzburg Landau equation

$$\Delta u + \frac{1}{\epsilon^2} u(1 - |u|^2) = 0,$$

when both u and x are two-dimensional. The physical motivations for studying this problem area arise primarily in the theory of superfluids and superconductors, where this equation is counted among the more simplistic of the models describing the behavior of the density of superconducting electrons or of the superfluid. In fact, the properties of solutions firmly established by this work reflect some experimental behavior of those materials.

The two-dimensional character of the mathematical theory given here is, in part, a reflection of the fact that vortices in, for example, superfluids are often filaments arranged in some kind of pattern, describable in terms of a planar cross-section in physical space. For reasons to be hinted at below, the mathematical theory in two space dimensions is considerably different from that in higher dimensions.

As is so often true in other fields as well, motivations from physical considerations for studying this problem are matched by strong motivations from a purely mathematical viewpoint. The latter come largely from the theory of harmonic maps. Consider a smooth simply connected domain $G \in R^n$ and functions u from G to S^{n-1}, taking on given boundary values $u(x) = g(x)$ for x on the boundary of G. If $n \geq 3$, there exist such functions with finite energy $\int_G |\nabla u|^2 dx$; this energy can be minimized in H^1, and the minimizers are called harmonic maps from G into S^{n-1} with the given boundary values. Such a map will have singularities when $d = \deg(g, \partial G) \neq 0$ and the local nature of the harmonic map near them is known.

This concept breaks down when $n = 2$, as there exist no such functions with finite energy when $d \neq 0$. There do exist functions which are analogous to harmonic maps in this case, however, and their local behavior is similar to that when $n \geq 3$. Part of the purpose of this book is to develop an analogous theory.

At least two approaches are naturally suggested. One is to perforate the domain G with a number of small holes, so that it is now multiply connected, and consider the analogous minimization problem on the perforated domain. It will have a solution whose (minimal) energy can be examined as a function of the radius, number, and locations of the holes. It consists of a term independent of the location of the holes which approaches infinity as the radius approaches zero, plus a bounded remainder. One can then (1) choose the locations of the holes so as to minimize this energy remainder, and (2) pass to the limit as the radius approaches zero (along a subsequence) to obtain a limiting function which is one analog of a harmonic map in this scenario.

A second approach would be to expand the range of the function u to all complex numbers (2-vectors) rather than just the unit circle S^1, but to add to the energy a term which penalizes the deviation of $|u|$ from 1. This penalty term is here taken to be $\int_G \epsilon^{-2}(1-|u|^2)^2 dx$ (the authors note that similar penalty functions would do as well). The new energy has a minimum achieved at some function u_ϵ, and one then studies this limit as $\epsilon \to 0$.

This book follows both approaches and shows that they lead to the same results. One suspects that the same is true for a much wider collection of "analogs". It is shown that the limiting solution, in either case, is a function with exactly d isolated singularities (if $d > 0$) and that each of these singularities is locally of the form $\frac{(z-a_i)}{|z-a_i|}$, where a_i is its location (we are denoting position in G by the complex number z). Moreover, the locations a_i are such as to minimize a certain renormalized energy, related to the remainder mentioned above.

Many other results are proved about the limiting function in this book, as well as properties of limits of solutions of the GL equation which are not minimizers of the energy functional. A rich list of open problems is supplied.

The book represents a remarkably complete and difficult theory of analogs of harmonic maps into S^1 in two dimensions in the scenario described above. It has already generated a great deal of interest among other mathematicians, and the theory is growing apace, including the simplification and extensions of some of the proofs presented in this book.

The fact that the theory of this PDE boundary value problem and the limiting singularities of its solutions was developed only recently is a surprise, given its great physical interest and potentially great mathematical interest. This attests to the difficulty of the subject. The book of Bethuel et al. is, of course, the definitive word, and it will be the starting point for anyone interested in pursuing this fascinating niche of mathematical physics.

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