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Free lattices, by Ralph Freese, Jaroslav Ježek, and J. B. Nation, Mathematical Surveys and Monographs, vol. 42, Amer. Math. Soc., Providence, RI, 1995, viii + 293 pp., \$65.00, ISBN 0-8218-0389-1

In this context, a lattice is an algebra with two idempotent, associative, commutative, binary operations, meet ($\wedge$) and join ($\vee$), which satisfies the absorptive laws

$$x \vee (y \wedge x) = x \quad \text{and} \quad x \wedge (y \vee x) = x.$$

A partial ordering of a lattice is defined by taking $x \leq y$ if $x \wedge y = x$. This partial order is such that $x \wedge y$ is the greatest lower bound and $x \vee y$ is the least upper bound of x and y. Conversely, a partial order such that any pair of elements has a least upper bound and a greatest lower bound is easily seen to define a lattice. The dual of a lattice is the lattice obtained by exchanging meet and join, or equivalently, by reversing the sense of the partial ordering of the lattice.

For example, the set of positive integers ordered by divisibility is a lattice which also satisfies the distributive law $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. The set of subgroups of a group, ordered by inclusion, is a lattice. The set of normal subgroups of a group is a modular lattice, i.e., a lattice satisfying the modular law: if $z \leq x$, then $x \wedge (y \vee z) = (x \wedge y) \vee z$. In logic, a Boolean algebra is a distributive lattice having an additional unary operation of complementation satisfying certain conditions. The set of theories in a given language forms a lattice ordered by inclusion.

Lattices are an important tool in modern algebra, and especially in universal algebra. In the same manner that one considers the automorphism group or endomorphism monoid of an algebra, the lattice of subuniverses and the lattice of congruences of an algebra are two more structures associated with an algebra that capture significant information about that algebra. For groups, these are essentially the lattice of subgroups and lattice of normal subgroups, respectively. The varieties of algebras of a given type form a lattice under inclusion which is dual to the lattice of equational theories of that type. In turn, methods of universal algebra have been applied to lattices or, in some cases, developed for and tested on the case of lattices.

A lattice $\mathbf{F}$ is freely generated by a subset $X \subseteq \mathbf{F}$ if X generates $\mathbf{F}$ and any map of X into a lattice $\mathbf{L}$ extends to a lattice homomorphism of $\mathbf{F}$ into $\mathbf{L}$. It follows that if $\mathbf{F}_1$ is freely generated by X_1 and $\mathbf{F}_2$ is freely generated by X_2 with $|X_1| = |X_2|$, then $\mathbf{F}_1$ and $\mathbf{F}_2$ are isomorphic. Given any set X, there exists a free lattice generated by X, and we can speak of the free lattice $\mathbf{FL}(X)$ generated by X as it is uniquely determined up to isomorphism. Since only the cardinality of the generating set matters, we also write $\mathbf{FL}(n)$ for the free lattice on a set of n generators. The notion of a free lattice is of course analogous to that of a free group and is an instance of the general idea of a free algebra in a variety of algebras.

The book *Free lattices* is a comprehensive treatise on the theory of free lattices, from Whitman's solution of the word problem for free lattices, to covers in free lattices and their relation to splitting lattices and the study of lattice varieties, to new results on the detailed structure of sublattices and intervals in free lattices. Along the way, the book presents many of its results in the more general setting of semidistributive lattices. Projective lattices and finitely presented lattices are examined. Efficient computer algorithms for working with lattices are presented and analyzed in a later chapter, as are results on AC term rewrite systems for varieties of lattices.

Although there are earlier results on free lattices, the subject received a substantial boost from two papers by Philip Whitman appearing in 1941 and 1942. Chapter I of *Free lattices* presents Whitman's solution to the word problem for free lattices, the canonical form for elements of a free lattice, and a number of results directly derivable from these. In a free group, each element is uniquely represented as a reduced product of generators and inverses of generators. We can understand elements of a free lattice in a similar manner, though the details are necessarily more complex. It turns out that it is easier to consider first an ordering of lattice terms defined by $s \leq t$ if $s^{\mathbf{L}} \leq t^{\mathbf{L}}$ for every lattice $\mathbf{L}$ where $t^{\mathbf{L}}$ denotes the interpretation of t in $\mathbf{L}$ and the inequality is required to hold for all values of the variables. This is a quasiordering of terms; terms s and t are equivalent if and only if $s^{\mathbf{L}} = t^{\mathbf{L}}$ in every lattice, i.e., if and only if $s \approx t$ is an identity of lattices. The free lattice $\mathbf{FL}(X)$ can be constructed by taking the set of terms in the variables X modulo this equivalence relation.

The relation $s \leq t$ can be syntactically characterized by considering cases according to whether s and t are variables, meets, or joins. If s and t belong to X, then $s \leq t$ holds if and only if $s = t$. If $s = s_1 \vee s_2 \vee \cdots \vee s_m$, then $s \leq t$ holds if and only if, for each $i, s_i \leq t$. Dually, if $t = t_1 \wedge t_2 \wedge \cdots \wedge t_n$, then $s \leq t$ holds if and only if, for each $j, s \leq t_j$. Somewhat less obviously, if $s = s_1 \wedge s_2 \wedge \cdots \wedge s_m$ and $t \in X$, then $s \leq t$ if and only if, for some i, $s_i \leq t$. Dually, if $s \in X$ and $t = t_1 \vee t_2 \vee \cdots \vee t_n$, then $s \leq t$ if and only if, for some $j, s \leq t_j$. These last two observations mean that the elements of X are both meet and join prime in $\mathbf{FL}(X)$. The remaining case is when s is a meet and t is a join. If $s = s_1 \wedge s_2 \wedge \cdots \wedge s_m$ and $t = t_1 \vee t_2 \vee \cdots \vee t_n$ and for some $i, s_i \leq t$, or for some $j, s \leq t_j$, then we have easily that $s \leq t$. The converse is Whitman's condition:

$$\text{(W)} \qquad \begin{array}{l} \text{If } s = s_1 \wedge s_2 \wedge \cdots \wedge s_n \leq t_1 \vee t_2 \vee \cdots \vee t_n = t, \text{ then,} \\ \text{for some } i, s_i \leq t, \text{ or else, for some } j, s \leq t_j. \end{array}$$

As presented in this book, the proof that Whitman's condition holds in free lattices is an elegant application of Alan Day's doubling construction.

The canonical form of an element of $\mathbf{FL}(X)$ is defined as a term of minimal length representing that element. Such a term is unique up to associativity and commutativity of meet and join. In fact, the book uses the convention that meets or joins of more than two terms may be written without unnecessary parentheses and such a term is shorter than a meet or join with extra parentheses (compare with reduced products representing elements in a free group). Thus, minimal length terms are unique up to commutativity. With this convention, the terms that are canonical forms of elements can be easily described. A variable is obviously a canonical form. A formal join $t = t_1 \vee t_2 \vee \cdots \vee t_n$ is in canonical form if and only if

(1) each t_i is either in X or is formally a meet;
(2) each t_i is in canonical form;
(3) $t_i \not\leq t_j$ for all $i \neq j$; and
(4) if $t_i = t_{i1} \wedge t_{i2} \wedge \cdots \wedge t_{im}$ is a formal meet, then for all $j, t_{ij} \not\leq t$.

A dual condition describes when a formal meet is in canonical form. This characterization also gives us a way of finding the equivalent canonical form for a given term: if condition (3) fails, then t_i can be deleted from t to give a join of fewer terms equivalent to t; if condition (4) fails, then $t_i \leq t_{ij} \leq t$, and so the term resulting from replacing t_i by t_{ij} in t is a shorter term equivalent to t.

We consider elements of the free lattice as represented by terms in canonical form. We thus define the canonical joinands of an element w to be the elements represented by the w_i if $w = w_1 \vee w_2 \vee \cdots \vee w_n$ is formally a join, and otherwise we take w itself as the only canonical joinand of w, when w is not canonically a formal join. We define canonical meetands of an element of the free lattice dually. An element in a free lattice whose canonical form is either a variable or formally a meet is join irreducible; it cannot be expressed as a nontrivial join. Each element in a free lattice is either join irreducible or meet irreducible. With these basic tools, one can show, for instance, that the automorphisms of a free lattice are induced by permutations of the generators and that $\mathbf{FL}(3)$ contains a sublattice isomorphic to $\mathbf{FL}(w)$.

Chapters II and III of the book develop the machinery of bounded homomorphisms, minimal join covers, semidistributive lattices, and splitting lattices leading up to the analysis of covers in free lattices. For u and v in a lattice $\mathbf{L}$, we say that u is covered by v and write $u \prec v$, if $u < v$ and there is no $w \in \mathbf{L}$ with $u < w < v$. In this case, we may also say v covers u, v is an upper cover of u, or u is a lower cover of v. A join irreducible element $w \in \mathbf{FL}(X)$ has a lower cover just in case it is completely join irreducible, i.e., it is not the least upper bound of the set of $v < w$. If w is completely join irreducible, its lower cover is unique and is denoted by w_*. Then w_* has a unique canonical meetand not above w, denoted $\kappa(w)$. This $\kappa(w)$ is completely meet irreducible, and its unique upper cover $\kappa(w)^*$ has w as the unique canonical join and not below $\kappa(w)$. Thus κ is a bijection from the set of completely join irreducible elements to the set of completely meet irreducible elements. Upper covers of a join irreducible element w are in correspondence with the completely meet irreducible canonical meetands of w. The book includes effective syntactic algorithms for recognizing completely join irreducible elements and for computing κ, algorithms which are useful for theoretical as well as experimental investigations.

The book presents two proofs of Alan Day's important result that finitely generated free lattices are weakly atomic; i.e., if $s < t$, then there exist u and v with $s \leq u \prec v \leq t$. Thus pairs $u \prec v$ abound in a finitely generated free lattice, and indeed they are key to an understanding of the detailed structure of free lattices. The book includes a complete analysis of finite intervals and of the connected components of the covering relation, i.e., the blocks of the equivalence relation generated by the covering relation. Sublattices of free lattices and projective lattices are considered. The finite sublattices of a free lattice are shown to be exactly the finite semidistributive lattices satisfying Whitman's condition (W). Totally atomic elements, those elements w for which every $v < w$ is below some lower cover of w and every $v > w$ is above some upper cover of w, are characterized; there are only finitely many in a finitely generated free lattice. New results on singular and semisingular elements are presented, and these lead up to an improved proof of this reviewer's result that every infinite interval in $\mathbf{FL}(X)$ contains a sublattice isomorphic to $\mathbf{FL}(\omega)$. These results, and a host of others including many new results, make up the largest part of the book.

Computational tools become increasingly important as one proceeds in the investigation of free lattices. The basic algorithms for testing inequalities between terms and computing canonical forms can be carried out by hand, and they are theoretically useful, but they can also be tedious. Finding completely join irreducible elements and computing κ is even more tedious. To produce enough experimental evidence to enable a reasonable conjecture about something like the characterization of totally atomic elements is a major computation. The last part of the book presents and analyzes a number of algorithms for computing with lattices collected from both the mathematical and computer science literature. Included are efficient algorithms for computing the joins and meets in a finite lattice from its order relation or from its covering relation and vice versa. Algorithms for computing congruence lattices of finite lattices, for determining simplicity and subdirect irreducibility of lattices, and for finding direct decompositions of lattices are presented. Of course, algorithms for free lattices are analyzed, but the case of finitely presented lattices is also included. Term rewrite systems are considered in the final chapter.

The book assumes some background in lattice theory. The first chapter would be a good introduction to free lattices in a course on lattice theory or algebra, and the first few chapters will certainly be of interest to anyone continuing in lattice theory or universal algebra. Researchers in lattice theory will find much of interest. The comprehensive nature of this book guarantees that those interested in further research in free lattices will find this book indispensable.

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