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Monotone dynamical systems: An introduction to the theory of competitive and cooperative systems, by Hal L. Smith, Mathematical Surveys and Monographs, vol. 41, Amer. Math. Soc., Providence, RI, 1995, x + 174 pp., $49.00, ISBN 0-8218-0393-X

The solutions of many sorts of differential equations preserve some type of order relation on initial data, boundary data, and/or inhomogeneous terms. It is often possible to exploit such order-preserving properties by finding solutions or constructing iteration schemes which converge monotonically to equilibria or provide bounds on other solutions. Methods of analysis based on monotonicity and comparisons have been around for a long time, but they have often been used in ad hoc ways to derive local results. In the last ten or fifteen years monotone methods have been integrated with the theory of dynamical systems to obtain results that are quite general and are frequently global in nature. As one might expect, the results for continuous time systems are stronger than those for discrete time or iterative systems, but a considerable amount can be said about both. Much of the recent interest in monotone dynamical systems arises from the study of mathematical models in biology and chemistry. For many of those models the traditional variational principles, symmetry arguments, and conservation laws of mathematical physics are not available. On the other hand, since biological and chemical models typically treat quantities such as population densities or concentrations of chemicals which are intrinsically positive, they usually preserve positivity of solutions and often have additional monotonicity or order-preserving properties.

Monotone dynamical systems are usually defined on ordered Banach spaces. (More general settings are possible but occur less frequently.) Recall that an ordered Banach space is a real Banach space X with a nonempty closed subset K known as the positive cone which has the following properties: $\alpha K \subseteq K$ for $\alpha \in \mathbb{R}_+$, $K + K \subseteq K$, and $K \cap (-K) = \{0\}$. An ordering is then defined by $x \geq y \Leftrightarrow x - y \in K$, with $x > y \Leftrightarrow x \geq y$, $x \neq y$. A dynamical system on X is monotone if it preserves the ordering of initial data. In practice it is often desirable to consider semidynamical systems and to distinguish between continuous and discrete time systems. A continuous time semidynamical system or semiflow is a continuous map $\varphi\colon X \times \mathbb{R}_+ \to X$ with $\varphi(x, 0) = x$ and $\varphi(x, t+s) = \varphi(\varphi(x, t), s)$ for all $x \in X$, $s, t \in \mathbb{R}_+$. The restriction to forward time allows the application of the theory to delay or partial differential equations which are not reversible with respect to time. For ordinary differential equations $\mathbb{R}_+$ may be replaced with $\mathbb{R}$, and then φ will be a flow. Continuous time systems are typically generated directly by differential equations. Discrete semidynamical systems can be defined similarly but with $\mathbb{R}_+$ replaced by $\mathbb{N}$. More typically discrete systems are defined by iterations of a continuous map $\psi\colon X \to X$ so that for $t \in \mathbb{N}$, $\varphi(x, t) = \psi^t(x)$. (The map ψ may be derived from differential equations, for example, via schemes such as Picard iteration.) In either case the system is monotone if $x \geq y \Rightarrow \varphi(x, t) \geq \varphi(y, t)$ whenever t is a positive real number (continuous case) or a positive integer (discrete case). Many of the most interesting results in the theory of monotone dynamical systems require some stronger sort of order-preserving property. In cases where the interior of the positive cone K is nonempty, one can define the ordering $x \gg y \Leftrightarrow x - y \in \mathrm{Int} K$. A semidynamical system is strongly monotone if $x > y \Rightarrow \varphi(x,t) \gg \varphi(y,t)$ for positive $t \in \mathbb{R}_+$ or $t \in \mathbb{N}$. The strongly monotone property is sufficient for most purposes, but it cannot be satisfied by those systems for which the natural cone has empty interior. Such systems occur in applications involving delay or partial differential equations. A weaker alternative hypothesis which is sufficient for many purposes is the strongly order-preserving property. A system is strongly order-preserving if it is monotone and whenever $x > y$ there are open neighborhoods U and V of x and y respectively such that $\varphi(U,t) \geq \varphi(V,t)$ for some positive t. If a continuous time system (i.e. a semiflow) is strongly monotone or strongly order-preserving and has the property that bounded semiorbits have compact closures, then in principle its dynamics are significantly constrained. In discrete time systems the constraints on the dynamics are somewhat weaker, but some of them are still present. In practice additional information is usually needed in either case to answer specific questions from applied mathematics.

A fundamental property of both continuous and discrete time monotone semidynamical systems is the order interval trichotomy. For $a, b \in X$ with $a < b$, the order interval bounded by a and b is defined as $[a,b] \equiv \{x \in X : a \leq x \leq b\}$. If a and b are equilibria for a strongly order-preserving semidynamical system φ and if $\varphi([a,b],t)$ is relatively compact for all strictly positive t belonging to $\mathbb{R}_+$ (or $\mathbb{N}$), then one of three alternatives must hold: (i) there exists an equilibrium c with $a < c < b$, or (ii) all semiorbits with initial data x satisfying $a \leq x < b$ converge to a as $t \to \infty$, or (iii) all semiorbits with initial data x satisfying $a < x \leq b$ converge to b as $t \to \infty$. In cases (ii) and (iii) there are full orbits (i.e. orbits which extend backward to $-\infty$ in t) connecting b to a or a to b respectively. The continuous-time version of this result was obtained by Matano [9]. The discrete time version was developed in the work of Hess and his collaborators; see [2]. To apply this result effectively, some knowledge of the stability properties of the equilibria a and b is often required. In many applications $\varphi(x,t)$ is differentiable in x for each t, and the linearization $D_x\varphi$ has positivity properties which permit the use (perhaps after a considerable amount of work) of results such as the Perron-Frobenius theorem for matrices or the Krein-Rutman theorem for linear operators. Those results imply the existence of a real eigenvalue which is larger than the real part of any other eigenvalue and whose eigenvector is positive. The positive eigenvector often can be used in conjunction with the monotonicity of the system to obtain local stability results.

Strongly monotone continuous-time systems, i.e. semiflows, have another striking fundamental property: If all semiorbits have compact closures, then convergence to the set of equilibria is in some sense "generic". Results of this sort were introduced by Hirsch [3–6]. The version given below is from [6]. Recall that the omega limit set of an initial point x is defined as $\omega(x) = \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} \varphi(x,s)}$. If φ is a strongly monotone semiflow such that all semiorbits of φ have compact closures, then $\omega(x)$ is contained in the set of equilibria of φ for all x outside of a countable union of nowhere dense sets. This version of the result is fairly general but not especially strong. If extra hypotheses are imposed on the semiflow, the underlying space, or the set of equilibria, then the conclusion can be strengthened in various ways. A related result asserts that periodic orbits cannot be stable. On the other hand, a construction of Smale shows that a strongly monotone flow on $\mathbb{R}^n$ can have an unstable $n-1$ dimensional invariant set upon which essentially any $n-1$

dimensional dynamics can occur. The question of exactly how and to what extent monotonicity restricts dynamics continues to be the subject of active research.

One class of equations which generate monotone flows are autonomous cooperative systems of ordinary differential equations on $\mathbb{R}^n$. The system

$$\dot{x}_i = f_i(x_1, \ldots, x_n), \qquad i = 1, \ldots, n, \tag{1}$$

is cooperative if $\partial f_i/\partial x_j \geq 0$ whenever $i \neq j$. For these systems the positive cone K is simply the usual positive orthant in $\mathbb{R}^n$. The monotonicity arises from comparison principles of the sort derived by Kamke [7]. If the Jacobian matrix $((\partial f_i/\partial x_j))$ is irreducible, the system will generate a strongly monotone flow. In the case where $\partial f_i/\partial x_j \leq 0$ for $i \neq j$ the system is said to be competitive. Competitive systems are not monotone with respect to the positive orthant. However, a two-dimensional competitive system will generate a flow which is monotone with respect to the cone $K = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 \geq 0, x_2 \leq 0\}$. More generally, a competitive system becomes cooperative if the direction of time is reversed. Cooperative and competitive systems occur in theoretical ecology as models of interactions between different species.

It is illuminating to compare the implications of symmetry and monotonicity for (1). If the system is of gradient type, i.e. if there exists $F(x_1, \ldots, x_n)$ such that $f_i = \partial F/\partial x_i$ for each i, the $\partial f_i/\partial x_j = \partial f_j/\partial x_i$ for all i, j so that the system is symmetric. The function $-F(x)$ acts as a Lyapunov function for the system and can be used to show that all bounded orbits have omega-limit sets which are contained in the set of equilibria. Furthermore, the Jacobian $((\partial f_i/\partial x_j))$ is symmetric so that its eigenvalues are all real. If the system is cooperative and the Jacobian is irreducible, then almost all bounded orbits have omega-limit sets contained in the set of equilibria [3–6]. Also, the Perron-Frobenius theorem implies that the eigenvalue of the Jacobian with the largest real part (which is the eigenvalue that determines linear stability at an equilibrium) is real. Thus, the implications of the monotone structure seem to be weaker than those of symmetry but somewhat similar in character. In applications where symmetry is not available monotonicity sometimes may be an acceptable substitute.

Monotone semiflows are generated by parabolic partial differential equations of the form

$$\begin{aligned} \frac{\partial u}{\partial t} &= Lu + f(u) \qquad && \text{on } \Omega \times (0, \infty) \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \partial\Omega \times (0, \infty) \end{aligned} \tag{2}$$

where Ω denotes a bounded domain in $\mathbb{R}^m$, L is a uniformly elliptic second-order partial differential operator with Hölder continuous coefficients and no zero order term, and $\partial u/\partial n$ denotes the normal derivative on $\partial\Omega$. (If L is a multiple of the Laplace operator, then (2) is called a reaction-diffusion equation.) Assuming that f is Lipschitz, the strong maximum principle as discussed in [11] implies that classical solutions u, v with $u(x, 0) \geq v(x, 0)$ on $\overline{\Omega}$ but $u(x, 0) \not\equiv v(x, 0)$ will satisfy $u(x, t) > v(x, t)$ on $\overline{\Omega}$ for $t > 0$. Standard existence and regularity results for parabolic equations imply that the reaction-diffusion equation (2) generates a semiflow on $C(\overline{\Omega})$ and that bounded semiorbits are precompact. Thus, (2) generates semiflow on $C(\overline{\Omega})$ which is strongly monotone with respect to the cone $K = \{u \in C(\overline{\Omega}) \colon u \geq 0$ on $\overline{\Omega}\}$. If the Neumann boundary condition $\partial u/\partial n = 0$ on $\partial\Omega$ is replaced with the Dirichlet condition $u = 0$ on $\partial\Omega$, then (2) generates a monotone semiflow on $C_0(\overline{\Omega})$ with the cone $K \cap C_0(\overline{\Omega})$, but there is a problem with strong monotonicity. To have $u \gg v$ requires that $u - v$ belong to the interior of the positive cone, but the interior of $K \cap C_0(\overline{\Omega})$ is empty in $C_0(\overline{\Omega})$. Thus the boundary condition interferes with strong monotonicity. This difficulty can be overcome by working in a more carefully chosen ordered Banach space. Alternatively, the restrictions on the ordered Banach space are reduced if strong monotonicity is replaced with the strong order-preserving property.

Systems of parabolic equations of the form

$$\frac{\partial u_i}{\partial t} = L_i u_i + f_i(u_1, \cdots, u_n) \text{ on } \Omega \times (0, \infty), i = 1, \ldots, n, \tag{3}$$

with L_i and Ω as in (2) will generate monotone semiflows under Dirichlet, Neumann, or Robin boundary conditions subject to the same sort of hypotheses on the functions f_i required for the ordinary differential equations in (1). Monotone and comparison methods have been widely used to study systems such as (3), but often without explicit reference to concepts from dynamical systems theory; see [8,10,15]. Other types of equations that can generate monotone semiflows include delay differential equations, e.g.

$$\dot{x}(t) = f(x(t), x(t - \tau)), \tag{4}$$

where $\tau > 0$ is a fixed constant. For delay equations the choice of positive cones is a rather delicate business, and to obtain general results it seems to be necessary to use cones with empty interior. That precludes strong monotonicity, so the use of arguments based on the strong order-preserving property is crucial. Delay equations and systems were treated from the viewpoint of monotone dynamical systems by Smith and Thieme [13,14].

Discrete time monotone dynamical systems can arise directly from difference equations or indirectly from iterations of maps associated with differential equations. Difference equations occur as models for populations with nonoverlapping generations. Monotone iteration schemes are widely used to show the existence of solutions to elliptic and parabolic partial differential equations and in numerical analysis. Another way that discrete time systems may be generated is by iteration of the Poincaré map (or time-T-map) for differential equations with T-periodic coefficients. Using the Poincaré map allows the treatment of periodic nonautonomous equations from the viewpoint of dynamical systems, but the price is that the systems are discrete. The arguments for monotonicity in the periodic case are typically the same as in the autonomous case, so the structural conditions for a system to be competitive or cooperative are the same as well. Many of the extensions of the theory of monotone dynamical systems to discrete time were made by Hess and his coworkers; those results are described and applied to periodic-parabolic equations in [2].

The essential ideas behind the theory of monotone dynamical systems were introduced around the beginning of the twentieth century. The basic concepts and methods of stability theory and the dynamical systems approach to differential equations were introduced by Lyapunov and Poincaré. In the original German version of *Methoden der Mathematischen Physik*, Courant and Hilbert refer to a 1912 note of Bieberbach describing a monotone iteration scheme for solving a nonlinear elliptic equation [1, p. 286]. (Apparently the reference was dropped in the later

English translation.) A number of the ideas and results that would contribute to the theory were introduced between 1925 and 1935. These include Birkhoff's work on dynamical systems, Lotka and Volterra's use of systems of differential equations to describe interacting biological populations, Hopf's strong maximum principle, and Müller and Kamke's comparison and monotonicity results for systems of ordinary differential equations. Over the next three decades there was a steady development of the infrastructure that would be required to give the theory of monotone dynamical systems its full power and scope: functional analysis, the theory of positive operators and semigroups of operators, and the modern theory of linear partial differential equations. From the mid-1960s through the 1970s there were three major trends which reinforced each other and set the stage for monotone dynamical systems theory. There was a surge of interest in nonlinear partial differential equations, especially reaction-diffusion equations [8, 10, 15]. There was also a surge of interest in dynamical systems and extensions of dynamical systems theory to infinite-dimensional settings [5]. Those two trends probably arose in part because of the inspiration of leading researchers and the availability of the necessary technical machinery, but in part because of the third trend. The third trend was a surge of interest in mathematical models in biology, chemistry, economics, and other areas of science outside of the more traditional realms of physics and engineering. Dynamical systems theory provided an intellectual viewpoint which could be used to describe complex phenomena and to organize a broad range of results and methods in nonlinear differential equations. Reaction-diffusion and delay equations could profitably be cast as infinite-dimensional semidynamical systems. Monotone and comparison methods were heavily used in the study of reaction-diffusion systems and were suggested by topics of applied interest such as the competition for resources by ecological populations. By the 1980s there had been enough interaction among the three trends that the time was right for a synthesis. The broadest unifying vision was articulated by Hirsch [3–6]. Significant independent insights are due to Matano [9]. Smith and his collaborators made important contributions to development of the continuous time theory and its application to delay equations and to biological models; many of those ideas are presented in the book under review. The discrete time theory was strongly influenced by the ideas of Hess [2]. Other contributors to the theory of monotone dynamical systems include R. H. Martin, P. Poláčik, J. Selgrade, and P. Takáč. The idea of interpreting periodic competition systems and periodic parabolic equations as monotone discrete time dynamical systems was foreshadowed in the work of A. C. Lazer.

Smith's book gives an overview of the current state of the theory of continuous time monotone semidynamical systems and illustrates the theory with applications to systems of ordinary and delay differential equations and reaction-diffusion equations. It updates the author's 1988 survey article [12] but is much broader in scope. Discrete time systems are not discussed, but these are treated in [2]. The approach to dynamical systems is quite classical and analytic in flavor. There are essentially no references to numerical computation or manifold theory, and chaos is mentioned only in passing. There are lots of arguments involving sequences of approximations, spectral theory, methods from the theory of differential equations, or basic operator theory. The general theory is presented in the first two chapters. The remainder of the book is devoted to exploring the implications of the theory and related methods in various contexts. The reader is led through the increasing levels of technical sophistication required by ordinary, delay, and partial differential equations. The discussions are illuminated with detailed analyses of specific systems. Most of the examples are taken from mathematical biology. A feature of the exposition which should appeal to applied mathematicians is the careful attention that is given to the details involved in applying the theory to the concrete examples. Problems arising in applied mathematics often seem to fall just outside the hypotheses of whatever general theory one wishes to use to study them. Bridging the gap between the theory and the application may require a shift in viewpoint. (Even the biggest fans of the maximum principle sometimes have to integrate by parts.) Smith shows the reader many of the tricks of the trade, and that makes his book more useful.

The book is written in a clear and attractive style, and errors are infrequent. Complete proofs are provided for the results that properly belong to the theory of continuous time monotone dynamical systems. Technical results from nonlinar analysis, operator theory, and the theory of differential equations are generally stated clearly, but the reader is often referred to the literature for details. The prerequisites for easy reading are a knowledge of the basic notions of dynamical systems theory and the theory of positive matrices and operators and some familiarity with the classical theory of differential equations. The exposition is thus at a level which should be accessible to advanced graduate students. On the other hand, the content is sufficiently up to date and substantial that even experts are likely to gain new insights. This book will be useful to students and researchers in dynamical systems, differential equations, and mathematical biology.

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Chris Cosner
University of Miami
E-mail address: gcc@paris-gw.cs.miami.edu