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Lectures on the geometry of Poisson manifolds, by Izu Vaisman, Progress in Mathematics, vol. 118, Birkhäuser, Basel and Boston, 1994, vi + 205 pp., \$59.00, ISBN 3-7643-5016-4

For a symplectic manifold M, an important construct is the *Poisson bracket*, which is defined on the space of all smooth functions and satisfies the following properties:

1. $\{f, g\}$ is bilinear with respect to f and g,
2. $\{f, g\} = -\{g, f\}$ (skew-symmetry),
3. $\{h, fg\} = \{h, f\}g + f\{h, g\}$ (the Leibniz rule of derivation),
4. $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ (the Jacobi identity).

The best-known Poisson bracket is perhaps the one defined on the function space of $\mathbb{R}^{2n}$ by:

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right). \tag{1}$$

Poisson manifolds appear as a natural generalization of symplectic manifolds: a *Poisson manifold* is a smooth manifold with a Poisson bracket defined on its function space. The first three properties of a Poisson bracket imply that in local coordinates a Poisson bracket is of the form given by

$$\{f, g\} = \sum_{i,j=1}^{n} \pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j},$$

where π_{ij} may be seen as the components of a globally defined antisymmetric contravariant 2-tensor π on the manifold, called a Poisson tensor or a Poisson bivector field. The Jacobi identity can then be interpreted as a nonlinear first-order differential equation for π with a natural geometric meaning. The Poisson tensor π of a Poisson manifold P induces a bundle map $\pi^{\#}: T^*P \to TP$ in an obvious way. The rank of the Poisson structure at a point $x \in P$ is defined as the rank of the bundle map at this point. If the rank equals the dimension of the manifold at each point, the Poisson structure reduces to a symplectic structure which is also called *nondegenerate*. In general, the image of the cotangent bundle under $\pi^{\#}$ defines a general distribution in the sense of Stefan and Sussmann, which is, however, completely integrable. Each maximal integral submanifold is symplectic and called a symplectic leaf. In other words, one can think of a Poisson manifold as a union of symplectic manifolds (usually of varied dimensions) fitting together in a smooth way, which is a result due to Kirillov [Kir] (see also Hermann [H]).

The classical Poisson bracket (see Equation (1)) was first introduced by Poisson in the early nineteenth century in his study of the equations of motion in celestial mechanics. In particular, Poisson noticed that the vanishing of $\{f, h\}$ and $\{g, h\}$ implies that $\{\{f, g\}, h\} = 0$. About thirty years later, Jacobi discovered the famous "Jacobi" identity which "explains" Poisson's theorem. In 1834, using the Poisson bracket, Hamilton found that the equations of motion could be written in the form: $\dot{q}_i = \frac{\partial H}{\partial p_i}$; $\dot{p}_i = -\frac{\partial H}{\partial q_i}$ (Hamilton's equations). Since then, Poisson brackets had been exploited for decades in the nineteenth century. However, at that time, all the brackets were nondegenerate and essentially came from the canonical symplectic structure on a vector space. The surprising breakthrough was made by Lie, who was the first one to look at degenerate Poisson brackets [Lie]. Lie considered as an example the simplest case: linear Poisson structures (i.e. Poisson brackets on a vector space such that the brackets of coordinate functions are linear functions). These are the famous Lie-Poisson structures, rediscovered by Berezin in the 1960s.

Nowadays, we think of a Lie-Poisson structure as living on the dual space $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$, where the bracket between linear functions is defined by exactly the same bracket relation as that of the Lie algebra. This is a simple but most illuminating example, which plays a profound role in Poisson geometry even today. For example, the symplectic leaves of the Lie-Poisson structure on $\mathfrak{g}^*$ are just its coadjoint orbits; their symplectic structure was discovered by Kirillov, Kostant and Souriau in the 1960s. These structures have played an important role in modern representation theory and geometric quantization theory. Although this latest development does not seem to have any predecessor in Lie's work, it was the degenerate Poisson structures that played a crucial role in Lie's discovery of his famous theory of Lie algebras and continuous groups (however surprising this may sound: see the historical note of Bourbaki in [Bo]).

In the twentieth century, Lie groups have become very popular in mathematics, but for quite a while, degenerate Poisson brackets were forgotten for some reason. Only as late as the middle of the 1970s has the study of Poisson geometry enjoyed renewed vigor. This is in part motivated by the discovery of many physical models arising from mechanical systems, especially systems with symmetry groups or constaints. For instance, Dirac discovered that mechanics with constraints is a theory with degenerate Poisson brackets [D]. Another example, perhaps even simpler, is the Euler equation of a free rigid body with a fixed point. The second important factor in reviving Poisson geometry is the considerable interest in completely integrable systems, where nondegenerate brackets (often infinite-dimensional) arise naturally. The increasing interest in infinite-dimensional Lie algebras, for example Gelfand-Fuchs theory, provides another strong motivation. Around the same time, many authors independently came across to the same object: general Poisson manifolds. Among them, Kirillov developed further the notion of local Lie algebras which encompasses that of Poisson structures [Kir], and Lichnerowicz introduced a precise definition of Poisson manifolds and initiated a systematic study of them [Lic].

Unlike symplectic manifolds, the understanding of the local structure of general Poisson manifolds is extremely difficult. A substantial contribution was made by Weinstein in 1983 in the famous paper [W1], where the local structure theory was established. In particular, Weinstein proved the splitting theorem, which asserts that an n-dimensional Poisson manifold is locally equivalent to the product of $\mathbb{R}^{2r}$ equipped with the canonical symplectic structure with $\mathbb{R}^{n-2r}$ equipped with a Poisson structure of rank zero at the origin. As a consequence, the local study of a Poisson structure can be reduced to the case where the rank at a point is zero, since the local structure of symplectic manifolds is essentially unique according to Darboux's theorem. In the same paper, Weinstein also formulated the well-known linearization problem, which has inspired much deep research (see the references cited in the book). According to the author's knowledge, however, the local structure of nonlinearizable Poisson structures has hardly been touched.

For global structures, both Poisson cohomology and Poisson homology are important. In 1977, Lichnerowicz introduced what he called "Poisson cohomology" of a Poisson manifold; when the Poisson structure reduces to a symplectic one, this Poisson cohomology coincides with de Rham cohomology [Lic]. Nowadays, Poisson cohomology can be understood as a kind of Lie algebroid cohomology like de Rham cohomology. I will start with Lie algebroids, a concept which may not be familiar to most readers, but which has nevertheless played an essential role in the recent development of Poisson geometry. For a given Poisson manifold P, the Poisson bracket on its functions extends naturally to a Lie bracket among all differential one-forms. For exact one-forms, it can be defind in an obvious way by letting $[df, dg] = d\{f, g\}$. Fur arbitrary one-forms, define

$$[\omega, \theta] = L_{X_\omega}\theta - L_{X_\theta}\omega - d(\pi(\omega, \theta)),$$

where $X_\omega = \pi^\#\omega$ and $X_\theta = \pi^\#\theta$. This formula was found independently by many authors, including Gelfand-Dorfman, Magri-Morosi, Koszul, Karasev and Coste-Dazord-Weinstein in the mid 1980s. Moreover, this bracket of differential forms satisfies the following two striking properties:

1. $[\omega, f\theta] = f[\omega, \theta] + ((\pi^\#\omega)f)\theta$,
2. $\pi^\#[\omega, \theta] = [\pi^\#\omega, \pi^\#\theta]$,

where ω and θ are one-forms on P and $f \in C^\infty(P)$.

All these properties make the cotangent bundle of a Poisson manifold P a special case of a more general object in differential geometry, called a *Lie algebroid* by Pradines (see [M] for more details on the theory). This was first noted by Coste-Dazord-Weinstein. More precisely, a Lie algebroid is a vector bundle A over a manifold P together with a bundle map $\rho\colon A \to TP$ (called the anchor) such that its space of sections is equipped with a Lie algebra structure satisfying the above two properties, with $\pi^\#$ being replaced by ρ. A Lie algebroid is a straightforward generalization of a Lie algebra. It may also be thought of as a "generalized tangent bundle" to the base manifold P since TP (with the natural bracket of vector fields and identity map as anchor) is obviously a Lie algebroid. In fact, for a Lie algebroid one has the basic ingredients to carry out the usual construction of differential calculus such as Lie derivatives, contractions, Schouten brackets, etc. Moreover, for any Lie algebroid, a cohomology can be defined which can be understood as an analogue of Lie algebra cohomology. For the tangent bundle Lie algebroid TP, its Lie algebroid cohomology is just the de Rham cohomology of the manifold. For the cotangent bundle Lie algebroid of a Poisson manifold P, it leads to the so-called Poisson cohomology of Lichnerowicz. Poisson cohomology groups of lower dimensions have significant meaning in Poisson geometry. In particular, the third one is related to the so-called deformation quantization. The geometric meaning of higher-dimensional ones is, however, not yet clear.

A few years later in 1985, Koszul discovered that on the space of differential forms of a Poisson manifold there exists a natural differential operator δ of degree -1 which is of square zero. Subsequently, Brylinski rediscovered this operator δ in connection with his study of noncommutative differential geometry and defined a homology group for a Poisson manifold under the name "canonical homology" [Br]. In the case of symplectic manifolds, δ corresponds to the exterior differential operator d (up to a sign) under the $*$-isomorphism, which is defined as a symplectic analogue of the star operator in Riemannian geometry. Later, both Poisson cohomology and Poisson homology were generalized to the algebraic context and applied to quantization by Heubschman.

Poisson manifolds are often considered as a classical analogue of (noncommutative) associative algebras. This viewpoint, which came from physics, has stimulated a lot of fruitful and active research in recent years, pioneered by Karasev-Maslov, Weinstein, and Zakrzewski, resulting in the theory of *symplectic groupoids*. The main idea is to find objects, between the geometric structures of classical mechanics and the algebraic structures of quantum mechanics, which are still geometric but which incorporate some quantum properties. Second, a certain quantization of these objects is expected to lead to deformation quantizations (or star products) of the underlying Poisson manifolds.

Symplectic groupoids are closely related to the notion of "symplectic realizations", which can again be traced back to Lie, who used the name "function groups". In [Lie], Lie defined a function group as a collection of functions of the canonical variables $(q_1, \dots, q_n, p_1, \dots, p_n)$ which is a subalgebra under the canonical Poisson bracket and generated by a finite number of independent functions $\varphi_1, \dots, \varphi_r$. In modern language, this means that $\mathbb{R}^r$ has a Poisson structure induced from the canonical symplectic structure $\mathbb{R}^{2n}$ in the sense that $\Phi = (\varphi_1, \dots, \varphi_r)\colon \mathbb{R}^{2n} \to \mathbb{R}^r$ is a Poisson map. Generally, a symplectic realization of a Poisson manifold P, as defined by Weinstein, is a Poisson map from a symplectic manifold V to P which is a surjective submersion (Weinstein called this a full symplectic realization [W1]). Lie proved that such a realization always exists locally for any Poisson manifold of constant rank. The local existence theorem of symplectic realizations for general Poisson manifolds was proved by Weinstein in 1983 [W1]. Subsequently, Weinstein found that for a general Poisson manifold there exists *globally* an essentially unique symplectic realization, which possesses a local *groupoid* structure (in general there is an obstruction for the existence of a global groupoid structure) compatible with the symplectic structure [W2]. This is what is now called a symplectic groupoid. At the same time, for general nonlinear Poisson brackets, the problem of the semiclassical approximation was under study by Karasev-Maslov in Russia. And Karasev independently discovered the structure of symplectic groupoids, which provides a proper phase space for the underlying nonlinear bracket [Kar]. In his study of classical pseudogroups, Zakrzewski in Poland also found symplectic groupoids independently.

The following illuminating example should be helpful in understanding the role of symplectic groupoids. For a Lie-Poisson structure $\mathfrak{g}^*$ (i.e. a linear commutation relation), its symplectic groupoid, which exists globally in this example, is T^*G with the canonical cotangent bundle symplectic structure. Here G is a Lie group integrating $\mathfrak{g}$, and the groupoid structure is based on the group structure of G. For a general Poisson manifold, considered as a certain nonlinear "Lie algebra", it is of course reasonable to expect that there exists some "group-like" object. This is what the symplectic groupoid is. Therefore a certain quantization theory of the symplectic groupoid should provide a tool for studying general nonlinear commutation relations, analogous to the use of Lie groups in the study of linear commutation relations.

To each (local) Lie groupoid one may associate a Lie algebroid as its infinitesimal invariant, similar to the connection between Lie algebras and Lie groups. It is not surprising that the Lie algebroid of the (local) symplectic groupoid of a Poisson manifold is in fact the cotangent bundle Lie algebroid of the Poisson manifold as discussed earlier. This coincidence provides another indication why symplectic groupoids are naturally associated with Poisson manifolds. Today, the theory of symplectic groupoids and their applications continues as a very active research area. At this point, I would like to recommend that the reader consult the recent book by Karasev and Maslov [KM] for its rich background and motivation (although the book may not be very accessible).

In the last fifteen years, a large class of interesting Poisson manifolds has emerged from the study of quantum group theory: Poisson-Lie groups. The notion of Poisson-Lie groups was first introduced by Drinfel′d in the early 1980s and studied by Semenov-Tian-Shansky and subsequently by Lu, Kosmann-Schwarzbach, Weinstein and many others. These Poisson structures are very important in applications: in quantum group theory and in integrable systems, for example. There are many interesting nontrivial examples of Poisson-Lie groups. For instance, every connected compact semisimple Lie group carries a nontrivial Poisson-Lie group structure as shown by Lu-Weinstein and Majid. Geometrically, Poisson-Lie groups provide us with a large class of interesting nonlinear (usually quadratic) and degenerate Poisson brackets, which make the theory of Poisson geometry more concrete. On the other hand, the study of Poisson groups has in turn brought numerous new ideas and questions which continue to stimulate a lot of current research (for example, the theory of Poisson groupoids and Lie bialgebroids: see [W3] and [MX]).

Although the study of the subject continues, as a branch of differential geometry, Poisson geometry already has many basics which are important for its future development and for applications. This is precisely what the book by Vaisman is trying to provide. Vaisman's book is devoted to the introduction of Poisson geometry. It begins with preliminaries on differential calculus such as the Schouten bracket and the Koszul formula, which provide the basic language for the study of Poisson manifolds. It then moves to local structure theory and to examples in Chapters 3 and 4. Poisson calculus and Poisson cohomology occupy most of Chapters 4 and 5.

The main subject of Chapter 7 is Poisson morphisms and related topics. The main technique is the so-called "coisotropic calculus" developed by Weinstein in [W3], which is now a powerful and frequently used tool in the study of Poisson manifolds, as its name suggests.

Chapters 8 and 9 of Vaisman's book deal with symplectic realizations and symplectic groupoids outlined above. He also treats *isotropic realizations*, which were studied by Dazord et al. in connection with the construction of symplectic groupoids of regular Poisson manifolds. Poisson-Lie group theory is treated in the last chapter of the book, where readers can find a clear and elementary introduction to the theory as well as basic terminology and many examples. The book also contains some of the author's own contributions to the subject, including his interesting results on Poisson cohomology based on spectral sequence and his work on cofoliations and biinvariant Poisson structures on Lie groups. In particular, the author develops a quantization theory, mainly at the level of prequantization, analogous to the usual geometric quantization for symplectic manifolds. The subject of quantization is mainly treated in Chapter 6.

I would like to consider *Lectures on the geometry of Poisson manifolds* as a textbook rather than as a monograph, although it contains some of the most recent research results. It assumes some familiarity with the theory of symplectic manifolds. I do not think this should cause any problem for the reader. There already exist a few (if not many) good books on symplectic geometry. Vaisman makes many remarks in various parts of the text, trying to bring the reader in touch with relevant material and recent research. The book does a very good job in basics: clearly explaining the definitions and carefully providing the proof of the presented results, most of which were previously only scattered over the literature. On the other hand, some of the basic and important results are only briefly mentioned: for example, the linearization problem (in particular, the important results of Conn [C]). Deformation quantization, one of the most important problems concerning Poisson manifolds, is only sketched (this topic is, of course, beyond the scope of geometry). I would like to refer interested readers to the two surveys [FS] and [W4] for the current state of this topic. Some background material, as well as motivations, are lacking in the book. I think that it would be helpful for the reader, while reading the book, to go back to the original papers mentioned in the bibliography for motivation and background. Some material, for instance Poisson calculus and Poisson cohomology, could be presented more coherently and intrinsically in terms of Lie algebroids, although ideas are already implicitly used there (the name of Lie algebroids is already mentioned at the beginning of Chapter 4).

Overall, it is a well-written and very enjoyable book. It will certainly be useful to graduate students and researchers who want to learn about the basics as well as the recent development of Poisson geometry. Personally, I am very grateful to Vaisman for his timely writing on such a rapidly developing subject. It is hoped that this book will not only popularize the subject, but even stimulate a great deal of further development of Poisson geometry.

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