BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 2, April 1996

Quadratic algebras, Clifford algebras, and arithmetic Witt groups, by A. J. Hahn, Springer-Verlag, Berlin and New York, 1994, xi +286 pp., \$34.00, ISBN 0-387-94110-X

Clifford algebras appear in myriad ways in astonishingly many areas of mathematics. For example, there is the wonderful generalization of a substantial portion of classical harmonic analysis to the setting of "Clifford analysis", which grew out of Dirac's use of Clifford-algebra valued operators to obtain linear differential operators which square to give the Schrödinger wave operator. (See the brief account of Clifford analysis in the recent book review [Mc] in this journal; for more extensive treatments, see the references cited there, such as the excellent book [GM].) Then there is the geometry of spin manifolds, which depend on the existence of the spin group as a double cover of the special orthogonal group; the spin group lives in the multiplicative group of the Clifford algebra. (See [LM] for a superb account of spin manifolds.) In representation theory, Clifford algebras are the source of the spin and half-spin representations of the special orthogonal group.

But the focus here is on Clifford algebras in algebra, and particularly in quadratic form theory. Let $q\colon V \to F$ be a nondegenerate quadratic form on a finite-dimensional vector space V over any field F, and let $B_q\colon V \times V \to F$ be the associated symmetric bilinear form defined by $B_q(v_1, v_2) = q(v_1 + v_2) - q(v_1) - q(v_2)$. To any such q on V there is an associated Clifford algebra $C(V)$ which is one of the fundamental invariants of q. The multiplication in $C(V)$ encodes the geometry determined by q on V. For, since $v^2 = q(v) \cdot 1$ in $C(V)$ for every $v \in V$, it follows that for $v_1, v_2 \in V$, $v_2v_1 = -v_1v_2$ iff $v_1 \perp v_2$ (with respect to B_q). Clifford algebras are very useful in the classification of quadratic forms, and they have been important ingredients in many of the significant advances in quadratic form theory over fields.

Besides quadratic form theory over fields, there is a rich classical theory of such forms over rings of integers in an algebraic number field, especially over $\mathbb{Z}$. More recently, there has been much interesting work on (sheaves of) quadratic forms over algebraic geometric varieties. See, e.g., [Kn], [Ar], [OPS], [PSu], [AEJ], [CTS] for a sampling of work in algebraic geometric quadratic form theory. This work demands knowledge of quadratic forms over the local rings of points on the variety and over the affine coordinate rings of affine subsets of the variety. The needs of the arithmetic and the geometric theories have led to the study of quadratic forms over a base which is an arbitrary commutative ring, not just a field. Here also, Clifford algebras play an important rôle. The book by A. Hahn is an engaging and well-written introduction to Clifford algebras and their associated structures for forms over a commutative ring.

To see what it is that is being generalized to rings, let us first recall a few fundamental properties of Clifford algebras over a field. Let $q\colon V \to F$ be a nondegenerate quadratic form over a field F, as above. By definition,

$$(*) \qquad C(V) \cong T(V)/I_q \ ,$$

where $T(V) = \bigoplus_{i=0}^{\infty} V^{\otimes i}$ is the tensor algebra of V, and I_q is the ideal of $T(V)$ generated by $\{v \otimes v - q(v) \cdot 1 \mid v \in V\}$. Then, $C(V)$ contains a copy of V, which is a generating set of $C(V)$ as an F-algebra. Indeed, if $\{w_1, \dots, w_n\}$ is a base of V, then $\{1\} \cup \bigcup_{k=1}^{n} \{w_{i_1} w_{i_2} \dots w_{i_k} \mid i_1 < i_2 < \dots < i_k\}$ is a base of $C(V)$ as an F-vector space. If the characteristic of F is not 2, the Gram-Schmidt process allows one to find an orthogonal base of V. The multiplication table for the corresponding base of $C(V)$ is then particularly easy to work out.

Relative to the grading on $T(V)$, the generators of I_q are sums of terms of even degree. Therefore, $C(V)$ inherits from $T(V)$ the structure of a $\mathbb{Z}/2\mathbb{Z}$-graded ring, obtained by separating out the terms of even and odd degree:

$$C(V) = C_0(V) \oplus C_1(V) \ ,$$

where $C_i(V) \cdot C_j(V) = C_{i+j}(V)$ (i, j taken mod 2). So, $C_0(V)$ is a subring of $C(V)$, but $C_1(V)$ is not a ring. If $\dim_F(V) = n$, then $\dim_F(C_0(V)) = \dim_F(C_1(V)) = 2^{n-1}$. Also, $C(V)$ has a canonical involution (i.e., an antiautomorphism $\tau \colon C(V) \to C(V)$ such that τ^2 is the identity) arising from the involution on $T(V)$ given by $v_1 \otimes \dots \otimes v_k \mapsto v_k \otimes \dots \otimes v_1$. Let

$$A(V) = \{z \in C(V) \mid zc_0 = c_0 z \text{ for all } c_0 \in C_0(V)\} \ ,$$

the centralizer of $C_0(V)$ in $C(V)$. Then, $A(V)$ is a 2-dimensional subring of $C(V)$, which, when the characteristic of F is not 2, is determined by the signed determinant d of q; this d is given by

$$d = (-1)^{n(n-1)/2} 2^{-n} \det(B_q(w_i, w_j)) \ ,$$

where $\{w_1, \dots, w_n\}$ is any base of V; so, d is uniquely determined only up to a nonzero square in F. The connection with $A(V)$ is given by $A(V) \cong F[X]/(X^2 - d)$. That is, if $d \notin F^{*2}$, where $F^* = F - \{0\}$, then $A(V) \cong F(\sqrt{d})$; but if $d \in F^{*2}$, then $A(V) \cong F \oplus F$. (When $\mathrm{char}(F) = 2$, either $A(V) \cong F \oplus F$ or $A(V)$ is a 2-dimensional Galois extension of F; it then determines the appropriate invariant of q analogous to the signed determinant. This invariant, discovered by Arf [A], takes its values in the additive group $F/\{c^2 + c \mid c \in F\}$ rather than in F^*/F^{*2}.) Regardless of $\mathrm{char}(F)$, the further structure of $C(V)$ depends on the parity of n. If n is even, $C(V)$ is simple (i.e., it has no nontrivial two-sided ideals) with center F. Since $C(V)$ has an involution, its class in the Brauer group $\mathrm{Br}(F)$ of central simple F-algebras has order 2 or 1. Also, $C_0(V)$ is simple with center $A(V)$, unless $A(V) \cong F \oplus F$; in the latter case, $C_0(V) = C_+(V) \oplus C_-(V)$, where $C_+(V)$ and $C_-(V)$ are isomorphic simple rings of dimension 2^{n-2} with center F. When n is odd, $C_0(V)$ is simple with center F, and $C(V) \cong C_0(V) \otimes_F A(V)$.

Let us assume now that $\mathrm{char}(F) \neq 2$, and let WF denote the Witt ring of anisotropic quadratic forms over F. This ring, whose structure embodies an enormous amount of information about the quadratic forms over F, has a distinguished maximal ideal IF consisting of the classes of all quadratic forms on even-dimensional vector spaces. Then $WF/IF \cong \mathbb{Z}/2\mathbb{Z}$ by the map sending a quadratic form q to its dimension mod 2 (where, by definition, $\dim(q) = \dim(V)$). Further, $IF/(IF)^2 \cong F^*/F^{*2}$, by the map sending an even-dimensional quadratic form to its signed determinant. Additionally, there is a well-defined homomorphism

$$e_2 \colon (IF)^2/(IF)^3 \to \mathrm{Br}_2(F) \ ,$$

mapping to the 2-torsion of the Brauer group of F; the map is given by taking the class of q to the Brauer class of its Clifford algebra. It was a very old question whether e_2 is surjective, and Pfister asked explicitly in [P] in 1966 whether e_2 is injective. These were among the major open questions in quadratic form theory for many years. They were finally settled in 1981, when Merkurjev gave his astounding proof, using K-theoretic methods, that e_2 is actually an isomorphism for every field F (of characteristic not 2). The surjectivity of e_2 amounts to saying that for every central simple F-algebra with involution, some size matrix algebra over A is isomorphic to a tensor product of quaternion algebras. Merkurjev's proof of surjectivity of e_2 was a major advance in the theory of algebras, as well as in quadratic form theory.

Milnor in [Mi] put the map e_2 in a more general perspective by pointing out that there could be maps $e_n\colon (IF)^n/(IF)^{n+1} \to H^n(F, \mathbb{Z}/2\mathbb{Z})$ (Galois cohomology) for every $n \geq 0$. It is well known that $H^0(F, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, $H^1(F, \mathbb{Z}/2\mathbb{Z}) \cong F^*/F^{*2}$, and $H^2(F, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Br}_2(F)$; the maps e_0 and e_1 correspond to the dimension parity and signed determinant isomorphisms noted above, while e_2 corresponds to the Clifford algebra map e_2 as above which Merkurjev proved to be an isomorphism. Conceivably, the posited map e_n could be an isomorphism, for every n. However, for general n, it is not even known whether e_n is well-defined. (It is known how e_n should be defined on the standard generating set of $(IF)^n/(IF)^{n+1}$.) Deep work of Arason, Merkurjev and Suslin, Jacob, and Rost has yielded proofs that e_3 is a well-defined isomorphism, as is e_4, and e_5 is well-defined.

The book under review concerns the generalization of Clifford algebras to the setting where the base is a commutative ring R. The typical setup is to have a nondegenerate quadratic form $q\colon M \to R$, where M is a finitely generated projective R-module. The Clifford algebra $C(M)$ can be defined just as before in $(*)$, and it will still have the canonical grading, $C(M) = C_0(M) \oplus C_1(M)$. But significant complications immediately arise. The analogue of the dimension of V over F is the *rank* of M as an R-module, which is a not necessarily constant function from the set of prime ideals of R to the set of nonnegative integers. (For any prime ideal $\mathfrak{p}$, $\mathrm{rk}(M)(\mathfrak{p})$ is the rank of the free module $M_{\mathfrak{p}}$ over the localized ring $R_{\mathfrak{p}}$ of R.) The variation in the value of the rank from prime to prime might seem to doom $C(M)$ to hopeless complexity, since the structure of $C(V)$ in the field case depends so much on the parity of $\dim_F(V)$. Fortunately, this difficulty can be overcome by observing that there is a canonical ring decomposition $R = R_1 \bigoplus R_2$ and correspondingly $M = M_1 \bigoplus M_2$, where M_i is a projective R_i-module and q maps M_i to R_i, with M_1 of strictly odd rank (at every prime ideal of R_1) and M_2 of strictly even rank. Then, $C(M) \cong C(M_1) \oplus C(M_2)$, and $C(M_1)$ (resp. $C(M_2)$) behaves much like $C(V)$ for $\dim(V)$ odd (resp. $\dim(V)$ even). Notably, $C(M_2)$ proves to be an Azumaya algebra over R_2. This is the analogue for a commutative base ring to an algebra over a field being central simple. Likewise, $C_0(M_1)$ is an Azumaya algebra over R. But, the further internal structure of $C(M)$ can be difficult to see, since even if M is a free R-module, there may be no orthogonal base of M with respect to B_q. Often, orthogonal bases will exist after scalar extension from R to various localizations of R. Thus, patching and descent techniques become significant, as they can give information about (M, q) and $C(M)$ from their localizations.

Regardless of the rank of M, let $A(M)$ denote the centralizer of $C_0(M)$ in $C(M)$. Then $A(M)$ is a separable quadratic (i.e., constant rank 2) algebra but need not be a free R-module. To elucidate what can occur with $A(M)$, the author gives an extensive discussion of separable quadratic algebras and free quadratic algebras, and the group structure on the family of isomorphism classes of such algebras. As in the field case, the structure of $A(M)$ is related to the discriminant of q. But, the discriminant is no longer an element of R^*/R^{*2} (R^* the group of multiplicative units of R); instead, there is a discriminant module of q, which is a projective R-module of constant rank 1 with a nonsingular symmetric bilinear form. Despite this and other complications, a nice theory of Clifford algebras over commutative rings emerges. There is, however, no analogue to Merkurjev's Theorem, for Parimala and Sridharan have given in [PS] an example of a commutative R for which the Clifford algebra map from strictly even rank quadratic forms over R to $\mathrm{Br}_2(R)$ is not surjective.

The main new feature in Hahn's development of the theory is his emphasis on what he calls special elements in $A(M)$. He shows that if M has strictly even or strictly odd rank, then $A(M)$ has a special element iff $A(M)$ is a free R-module (of rank 2); when this occurs, $A(M) \cong R[X]/(X^2 - aX - b)$, for $a, b \in R$ with $a^2 + 4b \in R^*$. When $A(M)$ has a special element, the connections between $A(M)$ and the discriminant module are easier to see, and the situation is closer to the classical setup over a field. While special elements do not always exist, Hahn shows that in the most important case, when M is a free R-module, $A(M)$ does have a special element.

The first twelve chapters of Hahn's book are in the nature of a graduate-level textbook, with detailed exposition and fairly complete proofs; this part is accessible to a reader having a little background in commutative algebra (or willing to dig into the background references Hahn provides). Much new material is covered in the exercises, for which the author gives extensive hints and references. The final three chapters are very interesting surveys of deeper results, for which a substantial further body of mathematics is assumed. Chapter 13 treats the Brauer group (of Azumaya algebras) of a commutative ring R, and the Brauer-Wall group of $\mathbb{Z}/2\mathbb{Z}$-graded Azumaya algebras over R, and the Witt group of quadratic forms over R, with special attention to the classical cases $R = \mathbb{Z}$ and $R =$ an algebraic number field. In Chapter 14, the focus is on Witt groups and associated objects for rings of algebraic integers. Here, a great body of classical number theory is assumed. The final chapter takes up topics in the analytic and geometric applications of Clifford algebras and their modules, including Dirac operators, spin manifolds, and isoparametric hypersurfaces. For this, the background needed is in analysis, topology, and differential geometry. The survey chapters are still highly readable, as Hahn gives clear statements of the results he is assuming and good references for omitted proofs. These chapters provide an excellent way of integrating the foundational material of the earlier part of the book with important subjects where it applies, without greatly lengthening the presentation or substantially duplicating material found elsewhere.

Hahn's book could be used successfully as a text or a collateral reference for a graduate course, or for self-study by a motivated graduate student, or by anyone with some curiosity about the subject. Many results are assumed without proof, more so than usual in an introductory graduate text, but good sources are always provided for the omitted proofs. This book provides a nice complement to the one by Knus [K], which covers much the same ground more comprehensively and is more like a research monograph. But for a first approach to the subject, Hahn's book is more accessible. I recommend this book highly.

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A. R. Wadsworth
University of California, San Diego
E-mail address: arwadsworth@ucsd.edu