BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 2, April 1996

The Riemann-Hilbert problem, by D. V. Anosov and A. A. Bolibruch, Aspects of Mathematics, Friedr. Vieweg, Braunschweig and Wiesbaden, 1994, ix + 190 pp., ISBN 3-528-06496-X

Systems of complex ordinary differential equations in the $(n+1)$-dimensional domain D are of the form

$$\frac{dy_i}{dz} = f_i(z, y_1, \dots, y_n), \quad i = 1, \dots, n, \tag{1}$$

with $f_1, \dots, f_n$ holomorphic in D. Given any point $(a, b_1, \dots, b_n) \in D$ there is a unique solution $y_1(z), \dots, y_n(z)$ of (1) with $y_i(a) = b_i, i = 1, \dots, n$. This solution is necessarily holomorphic in some neighborhood of a in $\mathbb{C}^1$. Such a local solution can be continued analytically to some Riemann surface X, giving rise to a solution of (1) defined for all $z \in X$. In general, very little can be said about X. However this is different if one considers systems (1) that are linear, meaning that

$$f_i(z, y_1, \dots, y_n) = \sum_{j=1}^{n} A_{ij}(z) y_j, \quad i = 1, \dots, n,$$

with $A(z) := (A_{ij}(z))$ a holomorphic $n \times n$ matrix; in this case (1) is usually written as

$$\frac{dy}{dz} = A(z)y, \tag{2}$$

y being the appropriate column vector. In this situation $D = S \times \mathbb{C}^n$ where S is some domain in the Riemann sphere $\overline{\mathbb{C}}$. For linear systems X turns out to be a covering surface of S and all solutions can be lifted to the universal covering surface $\widetilde{S}$ of S. For this reason the solutions of (2) are considered to be holomorphic functions on $\widetilde{S}$. It is convenient to consider the associated matrix equation

$$\frac{dY}{dz} = A(z)Y \tag{2'}$$

where Y is an $n \times n$ matrix. Y is a solution of (2′) if and only if the columns of Y are solutions of (2). Y is called a fundamental system of solutions precisely when its columns are linearly independent; this is equivalent to Y being a holomorphic map $\widetilde{S} \to GL(n, \mathbb{C})$. For two fundamental systems Y_1 and Y_2 the product $Y_1^{-1} Y_2$ is constant, whence $Y_2 = Y_1 C$ for some $C \in GL(n, \mathbb{C})$. In particular, if σ is a deck transformation of $\widetilde{S} \to S$ and Y is a fundamental system of (2′), then $Y \circ \sigma$ is a fundamental system of (2′) and thus there is a unique $\chi(\sigma) \in GL(n, \mathbb{C})$

with $Y = (Y \circ \sigma)\chi(\sigma)$. The resulting map χ from the group Δ of deck transformations of $\widetilde{S} \to S$ to $GL(n, \mathbb{C})$ is a group homomorphism and is called the monodromy representation associated with Y. Different fundamental systems of (2′) give rise to mutually conjugate monodromy representations. Conversely if χ_1 is the monodromy representation associated with the fundamental system Y_1 of (2′) and $\chi_2 : \Delta \to GL(n, \mathbb{C})$ is a homomorphism conjugate to χ_1, then χ_2 is the monodromy representation associated with an appropriate fundamental system Y_2 of (2′). Hence (2′) gives rise to a well-defined class of mutually conjugate monodromy representations $\Delta \to GL(n, \mathbb{C})$ which is called the monodromy of (2′).

Hilbert's 21st problem deals with linear systems (2) for which the matrix $A(z)$ is holomorphic in $\overline{\mathbb{C}} \setminus \{a_1, \ldots, a_N\}$ and has a pole of order 1 at a_j, $j = 1, \ldots, N$. Such a system is called a Fuchsian system of differential equations. One shows easily that (2) is Fuchsian if and only if there are $n \times n$ matrices $B_1, \ldots, B_N$ with entries from $\mathbb{C}$ such that

$$A(z) = \sum_{j=1}^{N} \frac{B_j}{z - a_j} \quad \text{and} \quad \sum_{j=1}^{N} B_j = v.$$

Hilbert's problem 21 is to prove that there always exists a Fuchsian system with given singularities $a_1, \ldots, a_N$ and given monodromy χ. There is a weakened form of Hilbert's 21st problem. Here, a Fuchsian system is sought that, in addition to the given singularities $a_1, \ldots, a_N$ and the given monodromy χ, has some extra singularities $a_{N+1}, \ldots, a_{N'}$. These extra singularities are called apparent singularities, and they are true singularities although all solutions of (2) remain single-valued in suitable neighborhoods of them. This weakened form should be called Hilbert's problem 21 with apparent singularities.

There is a companion problem that has to do with the nature of the singularity of $A(z)$ at a_j. While Hilbert's 21st problem requires a Fuchsian singularity, that is a pole of order 1, at a_j there is the more general notion of regular singularity of (2) at a_j. Loosely speaking, regular singularity of (2) at a_j means that every solution of (2) grows at most as some polynomial in $|z - a_j|^{-1}$ as $z \to a_j$ (for a precise definition see [AB], p. 8). The notion of regular singularity makes sense for nth order linear differential equations as well; the analog of Fuchsian singularity for n order linear differential equations is that for each $i = 1, \ldots, n$ the coefficient of $y^{(n-i)}$ has a pole of order $\leq i$ at a_j. While these two notions are equivalent for linear differential equations, this is not the case for systems of linear differential equations. However, as mentioned before, for linear systems each Fuchsian singularity is a regular singularity. Calling (2) a regular singular system if each singularity is a regular singularity, the companion problem to Hilbert's problem 21 is to prove that there always exists a regular singular system with given singularities $a_1, \ldots, a_N$ and given monodromy χ. This problem has been answered in the affirmative sense by Plemelj (see [P]) for $S = \overline{\mathbb{C}} \setminus \{a_1, \ldots, a_N\}$ in 1908 and by the reviewer ([R]) for $S = X \setminus \{a_1, a_2, \ldots\}$ where X is an arbitrary, compact or non-compact, Riemann surface and $\{a_1, a_2, \ldots\}$ a set without accumulation points on X.

Positive solutions to Hilbert's problem 21 were obtained in the following cases:

(i) $n = 2$ and N arbitrary (Dekkers, [D]) ;

(ii) n and N arbitrary, and for some small circle α_j around one of the points $a_j, j = 1, \ldots, N, \chi(\alpha_j)$ is semisimple (Plemelj, [P]);

(iii) n and N arbitrary, and $\chi(\alpha_1), \ldots, \chi(\alpha_N)$ sufficiently close to the unit matrix (Lappo-Danilevskii, [L]);
(iv) n and N arbitrary, and χ irreducible (Bolibruch, [B1], [B2]; Kostov, [K1], [K2]).

However, in 1989 the second author found a negative solution to Hilbert's 21st problem.

In Chapter 2 this counterexample is discussed. A certain non-Fuchsian system with $n = 3$ and $N = 4$ (3 of the singular points are Fuchsian, while the the 4th one is a pole of order 2) is explicitly given, and it is shown that there is no Fuchsian system with the same 4 singularities and the same monodromy. The main tool is Levelt's necessary and sufficient condition for a regular singularity to be Fuchsian.

Chapter 3 presents a proof of Plemelj's affirmative answer to the companion problem of Hilbert's 21st problem. This is done by constructing the principal $GL(n, \mathbb{C})$-bundle $P \to S$ associated with the given monodromy, as well as the corresponding vector bundle $E \to S$ and extending them to a holomorphic principal $GL(n, \mathbb{C})$-bundle $R \to \overline{\mathbb{C}}$ resp. a holomorphic vector bundle $G \to \overline{\mathbb{C}}$ in such a manner that global meromorphic sections of the latter bundles have polynomial growth at each $a_j, j = 1, \ldots, N$. Next the Birkhoff-Grothendieck theorem is proved. It says that each holomorphic vector bundle over $\overline{\mathbb{C}}$ is direct sum $\bigoplus_{i=1}^{n} \mathcal{O}(1)^{k_i}$ of powers of the holomorphic line bundle $\mathcal{O}(1)$ and that the exponents $k_1, \ldots, k_n$ are unique up to permutation. The chapter contains an elementary proof of this theorem. A simple application leads to Dekkers' result (i) and to Plemelj's theorem (ii).

The topic of Chapter 4 is irreducible monodromy, and after some technical preparations the theorem stated in (iv) is obtained.

Calling a non-increasing sequence λ of n integers admissible, the first highlight of Chapter 5 is the theorem that a representation χ is the monodromy of some Fuchsian system if and only if there is an admissible sequence λ such that the extension $R^\lambda \to \overline{\mathbb{C}}$ (for definition see [AB], p. 89) of $P \to S$ is holomorphically trivial. Next, reducible monodromies and regular singular systems are studied which leads to additional counterexamples to Hilbert's 21st problem. Moreover it is shown that any representation χ is a subrepresentation (quotient representation) of some representation for which Hilbert's problem 21 has a positive solution. The chapter concludes with some results on representations that cannot be obtained as the monodromy representation of some Fuchsian system.

Chapter 6 offers a complete description of all representations that cannot be realized as the monodromy representation of some Fuchsian system in the case $n = 3$. It is also shown that for $n = N = 3$ Hilbert's 21st problem is always solvable. Furthermore the authors prove that for any $n \geq 3$ and any points $a_1, \ldots, a_N$ with $N \geq 4$ there is a representation χ for which there is no Fuchsian system with singular points $a_1, \ldots a_N$ and monodromy representation χ. Finally there is the following instability result. If $n = 3$, then for any set $a_1, \ldots, a_N$, any representation χ, and any $\epsilon > 0$ there are points $a'_1, \ldots, a'_N$ with $|a_j - a'_j| < \epsilon$ for all $j = 1, \ldots, N$ such that there is a Fuchsian system with singularities $a'_1, \ldots, a'_N$ and monodromy representation χ.

The main result of Chapter 7 is that for any Fuchsian equation on $\overline{\mathbb{C}}$ there is a Fuchsian system with the same singular points and the same monodromy. The chapter ends with numerous examples.

The book is well written, although some phrases are a bit awkward, and is enjoyable to read. It is warmly recommended to advanced graduate students, to the mathematical community at large, and to the specialists.

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