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Nonlinear problems of elasticity, by Stuart S. Antman, Appl. Math. Sci., vol. 107, Springer-Verlag, Berlin and New York, 1995, xviii + 750 pp., $59.95, ISBN 0-377-94199-1

Elasticity theory is the central model of solid mechanics. Properly formulated, it gives rise to formidable nonlinear problems whose understanding is in many cases beyond the reach of present-day mathematics. Nevertheless, the last quarter-century has seen substantial advances in this understanding, due largely to the development and application of new methods of nonlinear analysis. An important part in this development has been played by Stuart Antman's pioneering studies of the existence and bifurcation of solutions for various rod and shell problems. His treatment of the theory in the monograph under review is thus of particular interest.

Perhaps the most famous nonlinear problem of elasticity is that of the buckling of a rod, and it well illustrates many of the difficulties of problem formulation and analysis typically encountered in the theory. Suppose we are given an initially straight cylindrical rod, which we try to compress by applying opposing forces to its two ends. If the rod is sufficiently thin and we push the ends together hard enough, the rod does not remain straight, but instead buckles into a curved configuration. This will happen however perfectly the rod is made and however careful we are to prevent asymmetries either in the composition of the rod or in the application of the forces. (Of course gravity will produce such an asymmetry, but this can be avoided partially by orienting the rod vertically, or almost completely by performing the experiment in a spacecraft.) Other similar buckling behaviour occurs for thin curved sheets under applied forces (sometimes accompanied by associated noises, as in the mistreatment of plastic coffee cups).

How can we model buckling of a rod mathematically? Suppose the rod has length L. We can identify the material points of the rod by their positions x in the open subset $\Omega = (0, L) \times D$ of $\boldsymbol{R}^3$, where the cross-section D is a bounded domain in $\boldsymbol{R}^2$. We call this undeformed configuration of the rod its *reference configuration*. We can thus describe any other configuration of the rod by the mapping $y : \Omega \to \boldsymbol{R}^3$ which takes each material point x to its deformed position $y(x)$.

If the rod is made from a material such as steel, wood or rubber, we can hope to model it as *elastic*, that is, as a material for which the (Piola-Kirchhoff) stress tensor $S(x)$ at the point $x \in \Omega$ in the configuration y depends only on the deformation gradient $Dy(x)$, which in rectangular Cartesian coordinates can be identified with the 3×3 matrix of partial derivatives $\frac{\partial y_i}{\partial x_j}(x)$. We write this dependence as $S = \sigma(Dy)$. If Σ is a smooth oriented surface passing through the point $x \in \Omega$ and having unit normal $N(x)$ there, then $S(x)N(x)$ gives the contact force per unit undeformed area acting at $y(x)$ across the deformed surface $y(\Sigma)$.

Of course buckling is a dynamic phenomenon, so that we are interested in motions of the rod described by one-parameter families of configurations $x \mapsto y(x, t)$ where the parameter t is the time. However, we shall first consider a static theory in which $y = y(x)$ is the only unknown. The governing partial differential equations are then given by the balance of forces

$$\operatorname{Div} \sigma(Dy) = 0, \tag{1}$$

where we assume for simplicity that the body-force (e.g. gravity) is absent. This is a system of three equations

$$\sum_{j=1}^{3} \frac{\partial}{\partial x_j} \sigma_{ij}(Dy(x)) = 0,$$

for $i = 1, 2, 3$ which have to hold for $x \in \Omega$. Since $\sigma = \sigma(A)$ is nonlinear, so is the system (1).

To complete the specification of the buckling problem, we need to prescribe suitable boundary conditions on y. This presents us with an awkward choice. On the lateral boundary $\partial\Omega_2 := (0, L) \times \partial D$ it is natural to require that the applied force vanishes, that is,

$$\sigma(Dy(x))N(x) = 0 \tag{2}$$

for $x \in \partial\Omega_2$, where $N = N(x)$ is the unit outward normal to the boundary $\partial\Omega$. But what should we specify on the rest of the boundary $\partial\Omega_1$ consisting of the two end faces of the rod $\{0\} \times D$ and $\{L\} \times D$? If we think about this, it becomes obvious that there are many different possible buckling experiments corresponding to different loading devices, these leading to different choices of boundary conditions. For example, we could specify equal and opposite *dead loads* parallel to the x_1 axis on the end faces, dead loads being those which maintain their direction and magnitude per unit undeformed area *however the rod deforms*. While such loads have the mathematical advantage of being conservative, it is hard to imagine how they could actually be applied, and they would lead to surprising behaviour not at all in keeping with the buckling phenomenon we are trying to study. For example, the rod could rotate through an angle π about the x_2 axis, leading to "compressive" loads becoming tensile! To rule out this behaviour, we could instead consider "live" or *follower* loads, in which the prescribed forces are normal to the *deformed* end faces. Such loads would also be difficult to apply in practice, and their nonconservative nature presents forbidding mathematical difficulties.

Instead of giving the forces on the end-faces, we could specify y there. For example, suppose that one end of the rod is welded to a rigid wall and the other to a rigid piston whose position is prescribed. While the boundary conditions corresponding to this realistic loading device are easy to write down, they have the disadvantage of not leading to a homogeneously compressed prebuckled state. As a compromise between reality and tractability we therefore instead consider the set of boundary conditions consisting of (2) and

$$\begin{aligned} y_1(0, x_2, x_3) &= 0, \\ y_1(L, x_2, x_3) &= \lambda L, \\ \sigma_{21}(Dy(0, x_2, x_3)) &= \sigma_{31}(Dy(0, x_2, x_3)) = 0, \\ \sigma_{21}(Dy(L, x_2, x_3)) &= \sigma_{31}(Dy(L, x_2, x_3)) = 0, \end{aligned} \tag{3}$$

for $(x_2, x_3) \in D$, where $\lambda > 0$, corresponding to end faces which are constrained to lie in the planes $\{y_1 = 0\}$ and $\{y_1 = \lambda L\}$ but are otherwise free to slide in these planes. A loading device that approximates these boundary conditions consists of compression of the rod by parallel lubricated plates, provided contact of the end-faces with the plates is maintained.

The classical view of buckling, due to Euler [8], is as an exchange of stability. As the displacement parameter λ passes through a critical value, a trivial solution (e.g. a uniformly compressed straight configuration of the rod) loses stability, and the rod moves to a new nontrivial stable configuration. Stability is a dynamic phenomenon, properly understood only through study of dynamical equations. However, we can draw on one key piece of dynamical information following from the second law of thermodynamics, the existence of a Lyapunov function for the governing dynamical equations, to motivate the *energy criterion* for stability, namely, that a configuration y is stable if it minimizes (at least locally) the total free-energy of the body. This is given by (at constant temperature, which we are assuming)

$$I(y) = \int_\Omega \varphi(Dy)\,dx, \tag{4}$$

where $\varphi = \varphi(A)$ is the free-energy function of the material. The corresponding Euler-Lagrange equations are the equilibrium equations (1), the stress being given by $\sigma(A) = D_A\varphi(A)$, where D_A denotes differentiation with respect to A. The function φ is required to be *frame-indifferent*, i.e.

$$\varphi(RA) = \varphi(A) \quad \text{for all} \quad R \in SO(3). \tag{5}$$

For simplicity, we suppose also that φ is *isotropic*, i.e. there are no preferred directions in the material as regards its mechanical response. This is expressed by the condition

$$\varphi(AQ) = \varphi(A) \quad \text{for all} \quad Q \in SO(3). \tag{6}$$

Suppose further that the reference configuration corresponds to an absolute minimum of φ, so that $\varphi(A) > \varphi(\mathbf{1})$ if $A \notin SO(3)$.

In order to analyze the exchange of stability we need a branch of trivial solutions to (1)–(3), and under some further hypotheses on φ (including *strong ellipticity,* a strengthened version of rank-one convexity defined below) such a branch $\bar{y}_\lambda, \lambda > 0$, can be shown to exist having the form

$$\bar{y}_\lambda(x) = (\lambda x_1, v(\lambda)x_2, v(\lambda)x_3), \tag{7}$$

where $v(\lambda) > 0$ with $v(1) = 1$. We now seek *critical values* of λ such that the mixed boundary-value problem obtained by linearizing the equilibrium equations and boundary conditions about $\bar{y}_\lambda$ has a nonzero solution, corresponding to a potential mode of instability of the rod. The greatest such critical value $\lambda_c < 1$ is a candidate for the value of λ at which buckling occurs. An explicit calculation of the critical values and corresponding linearized solutions is feasible in special cases (see [7] for rectangular and circular rods, where the issue of whether the mode corresponding to λ_c actually corresponds to buckling rather than barrelling is addressed; corresponding 2D calculations can be found in [6, 18]). However, little is known for more general cross-sections.

What can we deduce rigorously about solutions to the nonlinear problem from such a linearized analysis? We would like to show, for example, that a branch of buckled solutions bifurcates from the trivial solution at $\lambda = \lambda_c$. But a serious obstacle is that in order to apply bifurcation theory to the quasilinear system (1) we seem to be obliged to work in a space of quite smooth functions, such as the Sobolev space $W^{2,p}(\Omega; \boldsymbol{R}^3)$ for $p > 3$, and then the required regularity properties of the linearized operators are in doubt because of the abrupt change in boundary conditions at $\{0\}\times\partial D$ and $\{L\}\times\partial D$. If this difficulty could somehow be overcome, we could then ask whether an exchange of stability takes place at λ_c between the trivial and bifurcating solutions. An important issue here concerns the different possible meanings of "local minimizer", depending on the norm used. For example, under suitable hypotheses the trivial solution $\bar{y}_\lambda$ for $\lambda_c < \lambda \leq 1$ can be shown to be a local minimizer of I in the space $W^{1,\infty}(\Omega; \boldsymbol{R}^3)$, but more natural norms are those of $L^q(\Omega; \boldsymbol{R}^3), 1 \leq q \leq \infty$, or $W^{1,p}(\Omega; \boldsymbol{R}^3), 1 \leq p < \infty$. Unfortunately there is no satisfactory theory of local minimizers in the multi-dimensional calculus of variations that can handle these weaker norms.

In seeking to understand the structure of the set of all solutions an important role is played by absolute minimizers of the energy. For example, it can in many cases be proved that the trivial solution $\bar{y}_\lambda$ ceases to be a local minimizer even in $W^{1,\infty}(\Omega; \boldsymbol{R}^3)$ for $\lambda < \lambda_c$; if the absolute minimum of I subject to (2), (3) is attained, we are then assured of the existence of nontrivial solutions. The question of the existence of configurations absolutely minimizing the functional I subject to mixed displacement-traction boundary conditions such as (2), (3) is a story both of success and failure of modern applied analysis. The existence of such configurations was proved by the reviewer [2] in 1977 using the direct method of the calculus of variations (to a large extent inspired by Antman's work) under the hypotheses that the free-energy φ is *polyconvex*, i.e. that $\varphi(A) = g(A, \operatorname{cof} A, \det A)$ for some convex function g and that φ satisfies a suitable growth condition, for example in the version [15],

$$\varphi(A) \geq c_0(|A|^p + |\operatorname{cof} A|^q) - c_1, \tag{8}$$

where $p \geq 2$, $q \geq 3/2$ and $c_0 > 0, c_1$ are constants. This class of φ covers a wide range of useful models of materials, for example, various widely used isotropic free-energy functions for natural rubbers. But this result is known to be far from optimal. The hypothesis of polyconvexity should be replaced by the much weaker condition of *quasiconvexity*, introduced by Morrey [13] in 1952; namely, that for any bounded open $E \subset \boldsymbol{R}^3$

$$\int_E \varphi(Dv)\, dx \geq \int_E \varphi(A)\, dx = \mathcal{L}^3(E)\varphi(A) \tag{9}$$

for all 3×3 matrices A, whenever v is Lipschitz with $v(x) = Ax$ in a neighbourhood of the boundary ∂E. Quasiconvexity plays a pivotal role in the multi-dimensional calculus of variations and in particular is necessary and sufficient for sequential weak lower semicontinuity of I, this semicontinuity being the basis of the direct method. It is even in a sense necessary and sufficient for the existence of minimizers (up to the addition of a lower order perturbation). Unfortunately this desired weakening of the polyconvexity hypothesis suffers from two serious unresolved difficulties as the current theory stands. First, there is no known way of adapting the methods of Morrey to treat elasticity theory; the method makes essential use of piecewise affine approximations, and no one knows how to handle these in the case of elasticity, where φ has the singular behaviour (highlighted by Antman in his early work)

$$\varphi(A) \to \infty \quad \text{as} \quad \det A \to 0+, \tag{10}$$

corresponding to the requirement that it takes infinite energy to compress an elastic body to zero volume. Second, even if such an adaptation were possible, it would be next to useless because no one really understands, or knows in general how to verify, the nonlocally defined quasiconvexity condition (9). In his 1952 paper, Morrey discussed whether or not quasiconvexity was equivalent to the still weaker condition of *rank-one convexity* that $t \mapsto \varphi(A + ta \otimes n)$ be convex for all A and vectors $a, n \in \boldsymbol{R}^3$. He thought not, but apparently was not so sure when writing his 1966 book [14], and the matter was only finally resolved in 1992 by the now famous counterexample of Šverák [16]. Although quasiconvexity is a mathematically natural hypothesis, stored-energy functions φ that are not quasiconvex are of interest because they can model materials that undergo phase transformations, and their study is a currently very active research area (see, for example, [3]).

Even for polyconvex free-energies little is known about the properties of the energy minimizers that are known to exist, in particular practically nothing about their smoothness. In fact it is not even known whether they satisfy the Euler-Lagrange equation (1). As usual, the problem is the singular behaviour (10) of φ; this precludes the application, for example, of the partial regularity theory of Evans [9].

Another issue over which care must be taken is that of *interpenetration of matter*. We want configurations y to be invertible on Ω, though not necessarily on $\bar{\Omega}$. Thus the rod must not intersect itself, but self-contact could occur. When there is self-contact, the boundary condition (2) needs modification. However, when minimizing I, we ignore (2), since it is a natural boundary condition for the variational problem, and one can then attempt to handle the invertibility problem using the device of Ciarlet & Necas [5].

We have seen that a satisfying treatment of buckling of a rod on the basis of nonlinear elastostatics is at present out of reach. To carry out a corresponding dynamic analysis would require further breakthroughs. One aim of such an analysis would be to study connecting orbits between the unstable trivial solutions and the buckled states of the rod, another to establish whether or not the dynamically stable equilibria are really those that are local energy minimizers in an appropriate norm. Unfortunately, there is no set of physically reasonable dynamical equations known for nonlinear elastic materials in 3D for which there is a theory of existence and qualitative behaviour of solutions of sufficient scope to answer such questions.

Are we making the buckling problem unnecessarily complicated by insisting on using 3D nonlinear elasticity? Most engineers would answer yes and would be satisfied by an analysis based on a *rod theory*, the approach used by Euler. According to such theories, and ignoring thermal effects, a configuration of a thin rod is represented by a mapping $y : [0, L] \to \boldsymbol{R}^3$, together possibly with other variables $d_k : [0, L] \to \boldsymbol{R}^3$ called *directors* (their number depending on the degree of sophistication of the theory) giving information about the deformation of cross-sections of the rod. In the case of statics the governing equations are ordinary differential equations, for which many of the obstacles encountered in the analysis of the 3D theory can be overcome. Using a simplified rod theory, the *elastica*, Euler was able to give a definitive treatment of planar buckling with explicit formulae for the compressive load P_c at which buckling occurs; for a uniform rod with hinged ends $P_c = EI(\pi/L)^2$, where E is the Young's modulus and I the second moment of area of the cross-section.

While rod theories provide a simpler route to modelling buckling, their use raises conceptual and mathematical issues of the type well known in other areas of physics where models at different levels are used to describe the same phenomena. Ideally, we would like theorems saying that in appropriate circumstances solutions to the 3D theory are well approximated by those obtained from a rod theory. Although there has been much interesting recent work in identifying various rod and shell models through postulated asymptotic expansions of solutions to the 3D theory, the validity of such expansions is rarely addressed. An exception, but in a very special situation, is the work of Mielke [12]. For buckling problems we would in particular like to know if the critical buckling displacements/loads as calculated from the 3D theory converge to those of a rod theory as the thickness of the rod goes to zero. Such results have been proved on the basis of explicit formulae in special cases (see [10]), but general results are lacking.

The monograph takes an unusual but logical path through the theory, in the opposite direction to that given above, and starting with a careful and comprehensive treatment of the theory of elastic strings that has no counterpart in the existing literature. First the equilibrium equations are derived following the method of Euler. Then the classical dynamical equations are obtained, the derivation being used to introduce key concepts such as hypotheses on constitutive equations, frame-indifference and weak solutions. The weak form of the equations is shown to be equivalent to the balance of linear momentum for arbitrary sub-bodies, anticipating the three-dimensional version of this key conceptual result (due to Antman & Osborn [1]) given later. Classical problems, such as the catenary, suspension bridge and velaria are discussed from a modern perspective, with the emphasis on rigorous results concerning the existence and multiplicity of solutions.

Next, a theory of planar equilibrium for elastic rods that can stretch and shear is developed, the theory of the elastica arising as a special case when the constraints of unshearability and inextensibility are imposed. Special problems such as inflated rings and the straight equilibria of whirling rods are discussed. Continuing a steady build-up towards more complex and testing problems, the buckling of rods, nonstraight equilibria of whirling rods and the buckling of arches are treated in turn. The necessary analytic tools of degree theory, bifurcation theory and the calculus of variations are introduced and explained as needed, with the help of useful appendices on linear and nonlinear analysis. There is an emphasis on careful problem formulation, especially concerning constitutive hypotheses and boundary conditions. A more general theory of rods with directors, the *special Cosserat theory*, is then developed and analyzed, followed by a corresponding theory for axisymmetric shells, the assumption of axial symmetry retaining the advantage that the governing equations in statics are ordinary differential equations.

The 3D theory makes its long awaited appearance half way through the book, happily in a self-contained treatment of continuum mechanics that includes a good discussion of kinematics, stress, constitutive equations, material constraints, isotropy and thermomechanics. The theory is then specialized to 3D elasticity, covering such topics as constitutive restrictions, semi-inverse solutions for compressible and incompressible bodies, universal deformations and motions, perturbation methods and the relationship of linear to nonlinear elasticity. Armed with 3D elasticity, Antman returns to the discussion of general rod and shell theories, via a projection method by which they may be regarded as being constrained 3D theories. This method has the advantage of clarifying the relationship between the constitutive equations for the rod and shell theories and those for the 3D theory. He discusses necking, Mielke's treatment of St. Venant's principle, buckling of plates, and the status of the von Karman equations with respect to the 3D theory. The book concludes with a chapter on nonlinear plasticity and one on dynamical problems that treats Riemann problems, travelling waves and blow-up of solutions. There is an excellent bibliography in support of the historical and other notes, and the book comes equipped with those other politenesses to the reader, a good index and careful proof-reading.

There are three other books available which adopt comparable mathematical approaches to nonlinear elasticity and which complement Antman's monograph. That of Ciarlet [4] is an excellent introduction to 3D elastostatics and methods for studying the existence of solutions. Valent [17] gives a careful discussion of methods of proving existence based on the implicit function theorem. Finally, the book of Marsden & Hughes [11] covers a broader range of topics, including dynamics, is notable for an original approach to constitutive equations using covariance, and will appeal to those comfortable with methods of differential geometry. Antman's monograph has little overlap in material with these books. A scholarly work, it is uncompromising in its approach to model formulation, while achieving striking generality in the analysis of particular problems. It will undoubtedly become a standard research reference in elasticity but will be appreciated also by teachers of both solid mechanics and applied analysis for its clear derivation of equations and wealth of examples.

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J. M. Ball
Heriot-Watt University
E-mail address: J.M.Ball@ma.hw.ac.uk