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Approximation procedures in nonlinear oscillation theory, by Nikolai A. Bobylev, Yuri M. Burman, and Sergey K. Korovin, Nonlinear Analysis and Applications, vol. 2, Walter de Gruyter, Berlin, 1994, xi + 272 pp., \$79.95, ISBN 3-11-014132-9

Since oscillatory processes play a significant and often crucial role in many areas of the physical and biological sciences, as well as in the mathematical modelling of economic and social phenomena, it is not surprising that oscillation theory has become an important part of contemporary applied mathematics. The theory is well developed, not only for ordinary differential equations but also for more general functional differential equations (compare the recent monographs [EKZ], [GL], and [LLZ]). However, the numerical analysis of oscillatory processes is still far from being complete.

The present book is devoted to periodical processes, the simplest kind of oscillatory processes. It focuses on the analysis of numerical methods for the approximate solution of nonlinear differential equations with periodic solutions, with particular emphasis on the topological principles underlying their convergence theory. Most of the main results are illustrated by applying them to systems of automatic control. There are, however, no concrete numerical applications or numerical illustrations.

In order to convey an idea of the organization of the book and the topics treated in it, it will be helpful to provide an outline of its contents. The list given below shows the headings of the three chapters and their principal sections only (each section contains anywhere from two to nine subsections).

Periodic solutions of the nonautonomous system

$$\frac{dx}{dt} = f(t,x) \qquad (x \in \mathbf{R}^N), \tag{1}$$

where $f(t,x)$ is T-periodic in t, continuous in all its variables, and locally Lipschitz in x, have (known) period equal to an integer multiple of T and can be characterized in terms of the shift operator U_t (also called the Poincaré-Andronov operator). Let $p = p(t,s,x)$ denote the (unique) solution $x(t)$ of (1) corresponding to the initial datum $x(s) = x$. If the solution $p(t,s,x)$ is defined for all $t \geq s$, then it is said to be extendible to $+\infty$. In this case there exists a (continuous) operator U_t (the shift operator) from $\mathbf{R}^N$ to $\mathbf{R}^N$ given by

$$U_t(x) := p(t,0,x)\ .$$

It is easy to show that the solution $x(t)$ of (1) is T-periodic if, and only if, the point $x_0 := x(0)$ is a fixed point of the operator U_t.

For the autonomous system

$$\frac{dx}{dt} = f(x) \qquad (x \in \mathbf{R}^N)\ , \tag{2}$$

the problem of finding periodic solutions is more difficult: here, the period is not known a priori. Let U_λ denote the shift operator over the time λ along the orbits (paths) of equation (2). To find the initial data that lead to periodic solutions for (2), one has to determine values of λ such that the operator U_λ has fixed points and then to find those fixed points: if $\lambda = \lambda_0$ is a value for which U_{λ_0} has a fixed point x_0, i.e. $U_{\lambda_0}(x_0) = x_0$, then

$$x_0(t) := U_t(x_0) \tag{3}$$

is a periodic solution of (2), and its period is λ_0. A (theoretical and numerical) difficulty arises from the fact that fixed points of the shift operator are in general not isolated. To see this, assume that the function given in (3) is non-constant, and consider the function $x_h(t) := x_0(t+h)$. This function is a solution of (2): we have

$$\frac{dx_h(t)}{dt} = \frac{dx_0(t+h)}{d(t+h)} = f(x_0(t+h)) = f(x_h(t))\ .$$

Also, $x_h(t+\lambda_0) = x_h(t)$; thus, $x_h(t)$ is a λ_0-periodic solution of the autonomous system (2). This implies that $x_h = x_h(0) = x_0(h)$ is, for any h, a fixed point of the operator U_{λ_0}. In other words, the entire curve $\ell :\ x_h = x_h(t)$ consists of λ_0-periodic solutions to (2).

In order to generate numerical approximations to T-periodic solutions of (1) and (2) (where T is not known a priori) or to equations arising in a single-circuit system of automatic control, e.g.

$$L\left(\frac{d}{dt}\right)x = M\left(\frac{d}{dt}\right)f(t,x)\ ,$$

with $L(p) := p^\ell + \sum_{j=1}^{\ell} a_j p^{\ell-j}$ and $M(p) := \sum_{j=0}^{m} b_j p^{m-j}$ $(\ell > m)$, the given problem is usually reformulated as an operator equation of the form

$$x = \mathcal{A}(x) \qquad (x \in E), \tag{4}$$

where $\mathcal{A}$ is a (completely continuous) nonlinear (integral) operator on an appropriate Banach space E. For example, the problem of finding T-periodic solutions of

$$\frac{dx}{dt} = Ax + f(t,x) \qquad (x \in \mathbf{R}^N), \tag{5}$$

where A is a constant matrix (with eigenvalues not equal to $2\pi ki/T,\ k \in \mathbf{Z}$) and $f(t,x)$ is T-periodic in t, is equivalent to finding T-periodic solutions of the integral equation

$$x(t) = \int_0^T G(t,s)f(s,x(s))\,ds\ . \tag{6}$$

Here, the kernel G is given by the Green's function associated with (the linear part of) the differential operator in (5),

$$G(t,s) = (I - \mathrm{e}^{AT})^{-1}\,\mathrm{e}^{A(t-s)} \qquad (0 \le t-s < T).$$

Typically, E will be some Banach space of functions $x(t)$ defined on $[0,T]$ and possessing Fourier series expansions

$$x(t) \sim a_0 + \sum_{k=1}^{\infty}\{a_k\cos(2\pi kt/T) + b_k\sin(2\pi kt/T)\}\ .$$

Let

$$E_n := \operatorname{span}\{1, \cos(2\pi t/T), \sin(2\pi t/T), \dots, \cos(2\pi nt/T), \sin(2\pi nt/T)\} \qquad (n \ge 1),$$

and denote by $P_n :\ E \longrightarrow E_n$ the projection operator which assigns to $x \in E$ its nth partial Fourier sum.

The *Galerkin method* generates an approximation $x_n \in E_n$ which is the solution of the equation

$$x_n = P_n\mathcal{A}(x_n). \tag{7}$$

The *collocation method* applied to (4) requires the approximation $x_n \in E_n$ to satisfy (4) at $2n+1$ distinct points $\{t_k\}$ in $[0,T]$,

$$x_n(t_k) = \mathcal{A}(x_n)(t_k), \qquad (k = 0, 1, \dots, 2n). \tag{8}$$

(In the present book (Chapter 3) the convergence analysis is restricted to equally spaced points.) If the basis of the space E_n is given by $2n+1$ trigonometric polynomials ψ_k possessing the canonical property

$$\psi_k(t_j) = \begin{cases} 1, & \text{if } j = k \\ 0, & \text{if } j \ne k, \end{cases}$$

then the collocation equation (8) can be written in the form (7), where P_n is now the interpolatory projection operator for the collocation points $\{t_k\}$.

In the *harmonic balance method* (introduced by N.N. Bogolyubov and N.M. Krylov in the mid-1930s) the approximation $x_n \in E_n$ is found directly from the given differential equation: in the case of (1), x_n is to satisfy the equation

$$\frac{d}{dt}x_n(t) = P_n f(t, x_n(t)) \ .$$

The last two methods, as well as the method of *mechanical (numerical) quadrature* and the *finite-difference method*, can be interpreted as (perturbed) Galerkin methods, and hence their convergence properties can be studied within the common theoretical framework of projection methods. However, it is more natural to analyze the method of numerical quadrature and the finite-difference method in appropriate "discrete" spaces. Two powerful (and closely related) approaches for doing this were suggested in the 1960s by P.M. Anselone and R.H. Moore, K.E. Atkinson, and others (the monograph [A] contains a comprehensive account of this theory), and by G. Vainikko (compare the collection of papers [V1] and the monographs [V2], [KV]). Many of the ideas leading to this theory of discrete convergence occur already in the contributions by I.P. Mysovskikh (1956), H. Brakhage (1960), and L.V. Kantorovich and V.I. Krylov (1962); they were complemented by those of F. Stummel and R.D. Grigorieff in the early 1970s.

Anselone and Moore made use of the notion of collectively compact operators for analyzing discrete approximations to the operator equations (4), (6), while Vainikko viewed the methods yielding such approximations as particular realizations of the *factor method*.

In the factor method the given equation (4) is approximated in factor spaces of a given Banach space E rather than in subspaces. Let E_n $(n \geq 0)$ be Banach spaces, and let $p_n : \ E \longrightarrow E_n$ be a linear bounded operator with the property that

$$||p_n x||_{E_n} \longrightarrow ||x||_E \quad \text{as } n \to \infty \quad (x \in E).$$

The operator p_n is called a connecting operator, inducing the notion of a $\mathcal{P}$-convergent sequence of approximating operators $\mathcal{A}_n$ for $\mathcal{A}$ (cf. (V1),(V2)). As an illustration, let $E = C[0,T]$ $(=: C)$, with standard norm, and let $E_n = \mathcal{E}_h$ be the space of mesh functions x_n defined on the mesh $\Pi_n := \{kh : \ k = 0, 1, \ldots, n\}$ $(h := T/n, \ n \geq 1)$; if $x_n = (x_{n0}, x_{n1}, \ldots, x_{nn})$, then its norm is given by $||x_n||_{\mathcal{E}_h} := \max_{0 \leq i \leq n} |x_{ni}|$. The connecting operators $p_n : \ C \longrightarrow \mathcal{E}_h$ are defined by $p_n x := (x(0), x(h), \ldots, x(T))$. The space $\mathcal{E}_h$ is isomorphic to the factor space C/F_n, where F_n denotes the space of functions $x(t)$ possessing zero values on Π_n. If the isomorphism is $\Phi : \ C/F_n \longrightarrow \mathcal{E}_h$, and if $q_n : \ C \longrightarrow C/F_n$ is a canonical mapping, then the connecting mappings are given by Φq_n.

M. Urabe in 1965 was one of the first to analyze the numerical application of the Galerkin method to autonomous systems with periodic solutions (see, e.g., Chapter 11 in [U]). The fundamental paper by L.V. Kantorovich in 1948 on the abstract treatment of projection methods for linear equations was followed two years later by the work of M.A. Krasnosel′skii who presented a convergence theory of Galerkin methods for general nonlinear operator equations (4); its principal topological tool was the rotation of completely continuous vector fields (the monograph [KZ] has become the standard reference for this theory and its applications).

The analysis of collocation methods for integral equations and more general (linear) operator equations dates back to the work of L.V. Kantorovich and G.P. Akilov (1959); it reached its maturity with the subsequent contributions of G. Vainikko from 1970 onwards.

Although, as mentioned before, the harmonic balance method dates back to the 1930s, its use for systems of automatic control with forced oscillations and the (a posteriori) error estimation are still the subject of considerable research activity, as shown by the recent work of, e.g., G. Vainikko and the authors of this book during the last six years.

Since about 1970 the first author, N.A. Bobylev, has also played a major role in the development of the theory of numerical quadrature methods for oscillatory problems (and—with M.A. Krasnosel′skii and others—in the application of affinity theory to systems of automatic control). The origins of the general theory of discrete convergence for integral operators may be found in the work of L.V. Kantorovich and S.L. Sobolev in the late 1940s and early 1950s.

In the book under review the general convergence theory of projection methods and factor methods and a detailed discussion of the underlying topological principles (degree of a mapping; rotation of completely continuous vector fields, in particular those given by $\Phi(x) := x - \mathcal{A}(x)$; fixed-point theorems) form its core. A large part of the presentation of these topics reflects the development of the theory and numerical analysis of operator equations in the former Soviet Union from the early 1930s to the late 1980s. These topological principles form the foundation of the error analyses, in Chapter 3, for the Galerkin, harmonic balance, and collocation methods, and for the methods of numerical quadrature and finite differences. Results are first given for nonautonomous systems, with single-circuit and multi-circuit systems of automatic control serving as illustrations. Particular attention is then paid to the more complex case of autonomous systems where the period T is not known a priori and where, as indicated earlier, we may have a manifold ℓ of T-periodic solutions (leading to degeneracy in the underlying vector field).

Often the topological criteria of convergence of a numerical procedure for approximating periodic solutions of nonlinear systems are difficult to apply because the direct evaluation (or estimation) of the rotation of the underlying completely continuous vector field is not possible. In such cases affinity theory furnishes a possible alternative: it reduces the evaluation of the rotation of the completely continuous vector field in infinite dimensions to the same problem in a finite-dimensional setting where the role of $\mathcal{A}$ in $\Phi(x)$ is now assumed by the shift operator $U_T(x)$. A concise description of affinity theory and its application to the derivation of effective convergence criteria and stability results for the harmonic balance method conclude Chapter 3.

The book is carefully written, and the organization and exposition of the material are quite attractive. Chapter 1 is essentially used to introduce the reader to the numerical methods for solving (nonautonomous) systems and to describe how these seemingly very different methods are related and fit into the general frameworks of Galerkin or factor methods. Chapter 2 is a more or less self-contained introduction to smooth manifolds, exterior differential forms, and the topological tools described above; it assumes that the reader knows the "basic facts of functional analysis" (Preface). While the convergence analyses in Chapter 3 are comprehensive and elegant, the authors do not deal with topics like computable error bounds and the question of optimal order. In my view, the authors also missed an obvious opportunity to elaborate on the connection between, and the relative merits of, the factor method and Anselone's method of collectively compact operator approximations.

The book would have benefitted considerably from some careful copy-editing (incorrect expressions such as variety instead of variant (p. 22,40); network instead of mesh or grid (p. 33,37); numeric instead of, or in addition to, numerical (p. 33,34,39); closing instead of closure (p. 107), to mention just a few examples, could have been avoided) and by a more helpful index. In spite of these minor criticisms the book is a timely and significant contribution to this important field; it contains many results on the theory of numerical methods for oscillatory problems not easily accessible elsewhere.

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Hermann Brunner
Memorial University of Newfoundland
E-mail address: `hbrunner@kean.ucs.mun.ca`