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Shafarevich maps and automorphic forms, by János Kollár, Princeton Univ. Press, Princeton, NJ, 1995, 199 pp., \$37.50, ISBN 0-691-04381-7

Broadly speaking, one of the primary tasks in the theory of compact Riemann surfaces is the construction of meromorphic functions. A powerful tool in this endeavor is the Riemann-Roch theorem which allows the construction of functions with partially prescribed zeros and poles, such that the number of prescribed poles exceeds the zeros by a fixed constant depending on the surface. To be more precise, choose a finite collection of points p_i and integers n_i. This data, called a divisor, is usually represented as a sum $D = n_1p_1 + n_2p_2 + \dots$. The sum of the n_i is the degree of D, which is denoted by $deg\, D$. The vector space $L(D)$ consists of 0 together with meromorphic functions which vanish to order at least n_i at p_i if $n_i \geq 0$, or which have poles of order at most $-n_i$ at p_i otherwise. The Riemann-Roch theorem gives a formula for the dimension:

$$dim\, L(D) = deg\, D + 1 - g + i(D)$$

where g is genus of the surface (the number of holes) and $i(D)$ is a nonnegative integer. Dropping $i(D)$ yields an inequality, called Riemann's inequality, which is already quite powerful. For example, this inequality guarantees the existence of a nonconstant meromorphic function with simple poles at any $g + 1$ given points. In a more modern approach, the divisor D would be replaced by a holomorphic line bundle such that $L(D)$ corresponds to the space of its holomorphic sections. The degree of D coincides with a topological invariant of the line bundle called its degree or first Chern number, and $i(D)$ can be interpreted as the dimension of a certain first cohomology group. This abstract approach gives the right point of view for higher dimensional Riemann-Roch theorems due to Hirzebruch and others [H]. One complication in the higher dimensional setting is that the "error term" $i(D)$ would get replaced by an alternating sum of dimensions of higher cohomology groups, so one does not even have a Riemann type inequality in general. Thus in these cases it becomes essential to have good criteria for the vanishing of these higher cohomology groups. In the case of Riemann surfaces, it is known that $i(D) = 0$ as soon as $deg\, D > 2g - 2$, and the general principle is that the higher cohomology should vanish when things are "sufficiently positive".

An alternative approach to the construction of functions on Riemann surfaces comes from the uniformization theorem. This theorem states that any simply connected Riemann surface is biholomorphic to either $\mathbf{CP}^1$ (the Riemann sphere), $\mathbf{C}$ or the unit disc Δ. Thus the universal cover $\tilde{X}$ of any Riemann surface X is biholomorphic to one of the above surfaces. Therefore X can be represented as a quotient of one of these surfaces by a discrete group Γ of holomorphic transformations. Meromorphic functions on the original surface correspond to Γ invariant functions on the universal cover, and these can be constructed as ratios of holomorphic functions which are "almost" invariant or automorphic. These are holomorphic functions satisfying the functional equation:

$$f(\gamma z) = J(\gamma, z) f(z), \quad \gamma \in \Gamma$$

where the J's (the automorphy factors) are nowhere zero holomorphic functions satisfying a cocycle condition:

$$J(\gamma_1\gamma_2, z) = J(\gamma_1, \gamma_2 z)J(\gamma_2, z).$$

For example, if Γ is a group of 2×2 matrices acting on Δ by fractional linear transformations, then a standard example of an automorphy factor is

$$J_m(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) = (cz + d)^{2m}.$$

Given an arbitrary holomorphic function F, one can construct an automorphic form at least formally as a Poincaré series:

$$\sum_{\gamma \in \Gamma} J(\gamma, z)^{-1} F(\gamma z).$$

Convergence requires some care, but it can be established if F is bounded and $J = J_m$ with $m \geq 2$ [S, p. 55]. Thus this approach works well when $\tilde{X}$ is the disk where a large supply of bounded holomorphic functions exist. By contrast, the classical theory of automorphic functions on $\mathbf{C}$ (usually called theta functions) has a rather different character. The automorphy factors $\{J(\gamma, z)\}$ determine a holomorphic line bundle back on X such that the automorphic forms represent holomorphic sections pulled back to $\tilde{X}$. For example, $\{J_m\}$ corresponds to the mth tensor power of the holomorphic cotangent bundle. This circle of ideas can be completed by using the Riemann-Roch theorem to compute, or at least estimate, the dimensions of various spaces of automorphic forms.

One of the goals of Kollár's book is to attempt a similar program when X is a complex smooth projective variety, i.e., when X is a complex submanifold of a complex projective space. Namely, the goal of the book is to use automorphic forms to construct holomorphic sections of line bundles on X; specifically sections of the canonical line bundle K_X, which is the highest exterior power of the holomorphic cotangent bundle, and its higher tensor powers. The dimensions of these spaces, called plurigenera, are the fundamental invariants of a smooth projective variety. Kollár devotes a few chapters to the classically studied case where the universal cover is a ball or a bounded open subset of $\mathbf{C}^d$. Plenty of sections of large powers of the canonical bundle can be produced using Poincaré series. In fact this method shows that X has general type, which means that there is a constant $C > 0$ and an integer $n_0 > 0$ such that there are at least Cn^d linearly independent sections of the nth power of K_X for all n divisible by n_0. Poincaré series do not give much information for small powers of the canonical bundle. In this case, Kollár is able to prove the existence of nonzero holomorphic sections of quadratic and higher powers of K_X by using an appropriate Riemann-Roch theorem of Atiyah [A] and a vanishing theorem of Andreotti-Vesentini [AV].

For arbitrary smooth projective varieties, a major difficulty is that the universal cover $\tilde{X}$ is not well understood. A couple of decades ago, Shafarevich asked whether this manifold would be holomorphically convex [Sh, p. 407]. An affirmative answer together with Remmert's reduction theorem [GPR, p. 229] would imply that the compact connected subvarieties of $\tilde{X}$ could be contracted to points so as to obtain a possibly singular analytic space $Sh(\tilde{X})$ without any positive dimensional compact subvarieties. There has been quite a bit of work on this problem (e.g. [Gu], [KR], [N]); however, it is far from solved. Rather than attempting to solve it, Kollár draws inspiration from it and uses it to motivate his notion of the Shafarevich map. The idea is that the fundamental group $\pi_1(X)$ would act properly discontinuously on $Sh(\tilde{X})$ (were it to exist), resulting in a surjective holomorphic map of algebraic varieties:

$$X \to Sh(\tilde{X})/\pi_1(X).$$

This is the prototype of the Shafarevich map. Its characteristic property is that a subvariety Z of X would get contracted to a point if its preimage in $\tilde{X}$ is a disjoint union of compact varieties, or equivalently, if the image of the $\pi_1(Z)$ in $\pi_1(X)$ is finite. Kollár relaxed this condition to the requirement that the Shafarevich map $X - - > Sh(X)$ is a rational or meromorphic map with connected fibers, such that the above property holds only for sufficiently general Z's. The existence of $Sh(X)$ was established in an earlier paper [K] (see also [C]) and is also discussed in the book under review. This variety is not quite unique since there is some freedom built into the definition. However any two candidates for $Sh(X)$ are birationally equivalent, which means that they have isomorphic fields of meromorphic functions.

The structure of the Shafarevich map for a compact complex torus or a projective variety covered by the unit ball in $\mathbf{C}^d$ is particularly simple: it is a birational equivalence (it would be an isomorphism but for the fact that $Sh(X)$ is only well defined up to birational equivalence). Kollár calls a variety with this property a variety with *generically large fundamental group*. An equivalent formulation is that X has generically large fundamental group provided that the fundamental group of any "very general" subvariety should have infinite image in $\pi_1(X)$. Additional examples of such varieties can be obtained by taking products of complex tori and projective varieties covered by the unit ball (ball quotients from now on). The ball quotients can be singled out from among all these examples by the fact that they are of general type. These examples are (conjecturally) fairly typical, as they should exhibit some features of the whole class. For example, Kollár conjectures that any smooth projective variety with generically large fundamental group can be "fibered" over a base which also has generically large fundamental group and is of general type such that the smooth fibers are complex tori. The ball carries nonzero bounded holomorphic functions, and in general, Kollár shows that a variety with generically large fundamental group has general type as soon as its universal cover carries nonzero top degree holomorphic forms with finite L^p norm. The proof is an adaptation of an argument of Gromov [G]. The condition of having general type is an asymptotic condition about plurigenera; the above results for ball quotients suggests that it should be possible to obtain lower bounds for smaller plurigenera for varieties possessing generically large fundamental groups. In the second to last chapter, Kollár establishes some key theorems along these lines, and this is certainly the high point of the book. This chapter is entitled "Existence of Automorphic Forms". However, unlike the case of ball quotients, the techniques used here come primarily from within algebraic geometry.

I believe that Kollár has written a powerful book, and not coincidentally, a fairly demanding one. However, the explanations are clear, and I think that more than half of it would be accessible to anyone who has mastered the basics of complex algebraic geometry (say, the first 200 pages of [GH]). In attempting to describe some of the basic themes of the book, I may have been guilty of misrepresenting it as a narrowly focused research monograph when in fact it is quite the opposite. There are a number of general sections, such as chapters 9 through 11 on vanishing theorems, which could be read independently of the rest of the book and which would be of interest to almost any algebraic geometer. But to read only those sections would be a shame, because one would be missing a great deal of wonderful mathematics.

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