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Lectures on differential Galois theory, by Andy R. Magid, University Lecture Series, vol. 7, Amer. Math. Soc., Providence, RI, 1994, xiii+105 pp., $35.00, ISBN 0-8218-7004-1

1. Introduction

Differential Galois theory has known an outburst of activity in the last decade. To pinpoint what triggered this renewal is probably a matter of personal taste; all the same, let me start the present review by a tentative list, restricted on purpose to "non-obviously differential" occurrences of the theory (and also, as in the book under review, to the Galois theory of *linear* differential equations in characteristic 0):

—In 1984, J.-P. Ramis [16] discovered that the classical Stokes phenomenon one encounters in the resummation of divergent series has a Galoisian nature and can therefore be viewed as the effect of a generalized monodromy operator.

—In 1986, F. Beukers, D. Brownawell, and G. Hekman [3] realized that the technical hypothesis upon which Siegel had based his classical generalization of the Lindemann-Weierstass theorem amounts to a simple condition on a differential Galois group. This rejuvenated his approach to the theory of E- and G-functions.

—In 1988, N. Katz (cf. [10]) started a new way of investigating Sato-Tate conjectures on exponential sums by relating the measure involved in the associated law to a differential Galois group.

—In 1990, P. Deligne [5] rewrote the fundamentals of tannakian categories. In this theory, Galois groups preexist Galois extensions. This enabled him to give a new construction of Picard-Vessiot extensions.

—Algebraic extensions of function fields are particular cases of differential extensions. In this way, differential Galois theory can contribute to our knowledge of finite subgroups of classical groups (see, e.g., M. Singer and F. Ulmer [18]).

—Difference equations too have their own Galois theory. A number of authors have recently applied it with success to the study of recurrence relations and of their q-analogues.

Thus, interest in differential Galois theory is no longer restricted to specialists. But (strangely enough for a theory with so much historical appeal), textbook introductions are rare. Kolchin's exhaustive book [12] covers a much broader area (including, for instance, the Galois theory of nonlinear differential equations), at the cost of a heavy machinery to set in place. Kaplansky's excellent and brisk introduction to the linear theory [8] has one drawback (more on this in §4). As for Kuga's delightful *Galois' dream* [14], it is in fact concerned with Fuchsian equations and their monodromy groups (a finer object for such equations than the differential Galois group). So this new textbook is welcome.

The core of the book is expressed in a remarkably concise way on the first page of its preface: "The structure of the differential Galois extension is a twisted form of the function field of the differential Galois group, with scalars the base differential field," and it is on this fact that the whole theory will be based. We now explain these terms and describe at the same time what differential Galois theory is about, under the light of the tannakian approach (cf. [5, 10, 1]).

2. The main characters

Classically, one starts with a differential system

$$(*) \qquad \partial Y = AY,$$

where $Y = {}^t(y_1, \dots, y_n)$ is the unknown vector of functions, A is a $n \times n$ matrix with coefficients in the field $K = \mathbf{C}(z)$ of rational functions, and ∂ denotes the usual derivation d/dz. Fix an ordinary point $\omega \in \mathbf{C}$ of the system $(*)$ (see §4 for a discussion on this choice), and let $U(z) = (u_{ij}(z); 1 \le i, j \le n)$ be the fundamental matrix of solutions of $(*)$ with initial condition $u_{ij}(\omega) = \delta^i_j$. Because of $(*)$, the extension $L = K(U)$ of K which the u_{ij}'s generate in the field M_ω of germs of meromorphic functions at ω is stable under differentiation by ∂. This enables us to define the *differential Galois group of* $(*)$ as the group

$$(1) \qquad G = \{\sigma \in \mathrm{Aut}(L/K), \sigma\partial = \partial\sigma\}$$

of automorphisms of the field L which commute with ∂ and induce the identity on K. The differential extension L/K is called a Picard-Vessiot extension for $(*)$.

Now let σ be an element of G. For any solution Y of $(*)$ in L^n, the vector $\sigma(Y) = {}^t(\sigma y_1, \dots, \sigma y_n)$ is again a solution of $(*)$, since

$$\partial(\sigma(Y)) = \sigma(\partial Y) = \sigma(AY) = \sigma(A)\sigma(Y) = A\sigma(Y).$$

Therefore, there exists an invertible matrix $\rho(\sigma)$ with coefficients in $\mathbf{C}$ such that $\sigma(U) = U\rho(\sigma)$, and one easily checks that the map $\sigma \to \rho(\sigma)$ is a faithful left representation of G, so that G may be viewed as a subgroup of $\mathrm{GL}_n(\mathbf{C})$. Just like its classical namesake, the object of differential Galois theory is to set up a dictionary between the subgroups of G and the intermediate differential extensions of L/K, thereby enabling one to translate the properties of G (and of its representation ρ) in terms of L (and of the differential system $(*)$ which gave rise to L).

A rather different definition of the differential Galois group of $(*)$ is given in [9]. Although not formally needed in the discussion which follows, we shall now recall it as a step towards a more adequate description of G itself. Write V for the K-vector space K^n and D for the differential operator of order 1 on V given by $DY = \partial Y - AY$. Note that on each of the classical constructions of linear algebra (by which we mean here the K-vector spaces E deduced from V by taking its dual V^*, direct sums and tensor products, and iterating these basic constructions), there exists a natural extension of $D = D_V$ to a differential operator $D = D_E$ of order 1 on E: for instance, if $E = \mathrm{End}_K(V) = V \otimes V^*$, the differential operator $D = D_E$ is given in matrix terms by the rule $D(P) = \partial P - [A, P]$. This is the Lie algebraic version of the fact that a K-automorphism g of V acts on each of the constructions of V (on $\mathrm{End}(V)$, the action of g is given matricially by $g.P = gPg^{-1}$). Finally, say that a K-subspace W of a construction E is a D-subspace of E if $DW \subseteq W$ (for instance, since we are in characteristic 0, we may view a symmetric power S^pV as a D-subspace of $\otimes^p V$). Now, consider *the list* $\mathbf{X}(V, D)$ *of all* D*-subspaces in all constructions of* V, and define with Katz [9] the differential Galois group of $(*)$ as the group

$$(2) \qquad G'_K = \{g \in \mathrm{Aut}(V/K), gW = W \text{ for all } W \text{ in } \mathbf{X}(V, D)\}.$$

A good thing about G'_K is that it is obviously canonically attached to $(*)$ (it is "intrinsic" [1]), whereas G required a choice of base point ω and Cauchy's existence theorem to be defined. Another good thing is that by its very definition, G'_K is a linear algebraic group. But it is a subgroup of $\mathrm{Aut}(V/K) = \mathrm{GL}_n(K)$, not, as $\rho(G)$ is, of $\mathrm{GL}_n(\mathbf{C})$! So, how can they be given the same name?

3. Enters a torsor

Let us first show that, via its representation ρ, G can be given a definition akin to G'_K. For every construction E of V, write E^∂ for the $\mathbf{C}$-vector space of solutions of $D_E Y = 0$ in $E(M_\omega)$, so that on V itself, V^∂ is the $\mathbf{C}$-subspace of $V(L) = L^n$ generated by the columns of the fundamental matrix U. The definition of D_E and the wronskian lemma show that if E is given by a certain construction over K (say, $E = \mathrm{End}(V)$), then E^∂ can be deduced from V^∂ by exactly the same construction over $\mathbf{C}$ (which therefore lives in $E(L)$): for instance, $(\mathrm{End}(V))^\partial = \mathrm{End}_{\mathbf{C}}(V^\partial) \subseteq \mathrm{End}_L(V(L))$. Furthermore, D_E induces a differential operator D_W on each subspace W of E in the list $\mathbf{X}(V, D)$, and the $\mathbf{C}$-vector space W^∂ of solutions of $D_W Y = 0$ is just $W(L) \cap E^\partial$.

With this notation in mind, consider the group (cf. [5], 9.2, [10], 2.1)

$$G_{\mathbf{C}} = \{\gamma \in \mathrm{Aut}_{\mathbf{C}}(V^\partial), \gamma W^\partial = W^\partial \text{ for all } W\text{'s in } \mathbf{X}(V, D)\}. \tag{3}$$

It is in fact a tautology that (the set of $\mathbf{C}$-points of) $G_{\mathbf{C}}$ coincides with (the image under the representation ρ of) G. Indeed, Definition (1) immediately implies that the elements of $\rho(G)$ stabilize the W^∂'s. Conversely, an element γ of $G_{\mathbf{C}}$ acts by K-linearity on the linear forms on the symmetric algebra of $(V^\partial \oplus \cdots \oplus V^\partial) \otimes_{\mathbf{C}} K$ (n factors), i.e., defines an automorphism $\gamma^\sim$ on the polynomial algebra $K[X_{ij}(1 \leq i, j \leq n)]$. But since the set of K-linear forms on a construction E which vanish on any given element Y in E^∂ is a D-subspace W of E^*, we deduce from (3) that $\gamma^\sim$ stabilizes the ideal of algebraic dependence relations over K satisfied by the coefficients u_{ij}'s of U; in particular, this action of γ may be specialized to an automorphism of the subring $K[U]$ of L and thereby defines an element σ of $\mathrm{Aut}(L/K)$. Moreover, σ commutes with ∂, simply because γ sends a solution to another one. Thus, σ lies in G, $\gamma = \rho(\sigma)$, and $G_{\mathbf{C}} = \rho(G)$.

In a sense, we have just recovered Galois' initial vision of his groups: consider only those permutations of the roots that respect all polynomial relations they satisfy over the base field ([7], Prop. 1.2°, p. 51). The outcome is that $G = \rho^{-1}(G_{\mathbf{C}})$ is endowed with a structure of an algebraic group over $\mathbf{C}$, and we may speak of its extension of scalars $G_K = G_{\mathbf{C}} \otimes K$ to K (which is, in fact, where we saw γ in the argument above).

Now consider the following subset of $\mathrm{Hom}_L(V^\partial \otimes_{\mathbf{C}} L, V \otimes_K L)$ (see [5], 9.2 to 9.6, [10], 2.3.2):

$$P(L) = \{p \in \mathrm{Isom}_L(V^\partial \otimes_{\mathbf{C}} L, V \otimes_K L), p(W^\partial) \subseteq W(L) \text{ for all } W\text{'s in } \mathbf{X}(V, D)\}. \tag{4}$$

Because all these D-subspaces W are defined over K, this is the set of L-points of an affine K-subscheme, say, P, in $\mathrm{Hom}_K(V^\partial \otimes_{\mathbf{C}} K, V)$. Furthermore, given p and q "in" $P, q \circ p^{-1}$ (resp. $q^{-1} \circ p$) is an automorphism of V (resp. $V^\partial \otimes K$) leaving each W (resp. $W^\partial \otimes K$) stable. Thus, we deduce from (2) (resp. (3)) and (4) that P is a left principal homogeneous space under G'_K (resp. a right one under G_K). This is the torsor, or rather bitorsor (cf. [4], 2.4.3), we promised. This structure immediately tells us that P is reduced and that P and G_K become isomorphic over the algebraic closure K^{alg} of K, i.e., that the algebras

$$K^{\text{alg}} \otimes_K K[P] \text{ and } K^{\text{alg}} \otimes_K K[G_K] = K^{\text{alg}} \otimes_{\mathbf{C}} \mathbf{C}[G_{\mathbf{C}}]$$

are isomorphic. Incidentally, it also gives an answer to the last question in §2: the K-algebraic groups G_K and G'_K are isomorphic over K^{alg}.

Finally, by its very definition, the fundamental matrix U represents an element of $P(L)$, and the argument on $\gamma^{\sim}$ above shows that the Zariski closure over K of that point contains its orbit under $\rho(G)$. Since $G_{\mathbf{C}}$ is K-dense in G_K, we infer that P is an irreducible variety over K, which admits U as a generic point. In other words, the affine algebra $K[P]$ coincides with the subring $K[U, (\det U)^{-1}]$ of L. Putting everything together, we have eventually established the algebra isomorphism

$$K^{\text{alg}} \otimes_K K[u_{ij}(1 \leq i, j \leq n), (\det(u_{ij}))^{-1}] \approx K^{\text{alg}} \otimes_{\mathbf{C}} \mathbf{C}[G_{\mathbf{C}}], \tag{5}$$

hence the isomorphism $K^{\text{alg}} \otimes_K L \approx K^{\text{alg}}(G)$ announced at the end of §1. Moreover, a short argument shows that a K-rational function on P which is fixed under G_K (or equivalently, under $G_{\mathbf{C}}$) must belong to K; for the fixed field of $L \approx K(P)$ under G, this translates as:

$$L^G := \{x \in L, \sigma x = x \text{ for all } \sigma \text{ in } G\} = K. \tag{6}$$

4. The book under review

As we said in §1, the isomorphism (5) forms the cornerstone of Magid's approach to the Galois correspondence. The results we described in §3 will be found in Chapter 4 of the book (algebraicity of the Galois group), Chapter 5 (structure of the Picard-Vessiot extension) and part of Chapter 3 (the fixed field). However, the proofs are given in the more algebraic language of cogebras, which makes our explicit introduction of P unnecessary (see [17] for a general perspective on cogebras).

A more serious difference concerns the construction of the Picard-Vessiot extension L/K, for which the a priori knowledge of such overfields as M_ω enabled me to cheat in §2. What we need to check in general is not only that there exists a differential extension L of K splitting $(*)$, but also that it contains no new constants and, finally, that it is well defined up to (a non-unique) isomorphism. Kaplansky's book ([8], p. 22) does not address the question of existence, referring instead to an early paper of Kolchin [11], which, at least formally, still appealed to these overfields! On the other hand, Kolchin's book contains a purely algebraic proof of the existence and uniqueness of L ([12], IV, §5, Corollary 2, and VI, §6, Prop. 13; cf. also [6], [2]). These questions are studied in Chapters 1, 2, and 3 of Magid's book, together with comments on infinite Picard-Vessiot extensions. (We refer to [5], 9.3, for a construction of L from the a priori knowledge of the torsor P.)

Combining (5) and (6), Magid easily derives in Chapter 6 the main theorem of differential Galois theory: the map $H \to L^H$ gives a dictionary between the algebraic subgroups of G and the differential extensions of K inside L, where a normal subgroup H corresponds to a Picard-Vessiot extension of K, with differential Galois group isomorphic to G/H. This reflects the fact that any representation of G is a subquotient of a construction of V^{∂} ([17], Prop. 8) and can thus be tracked back to a subquotient of a construction of (V, D).

The book closes with a chapter on the differential analogue of the inverse problem in Galois theory, when the base field K is $\mathbf{C}(z)$, viz.: given an algebraic subgroup H of $\mathrm{GL}_n(\mathbf{C})$, find a differential system $(*)$ over K whose differential Galois group

G is isomorphic to H. Various types of answers to this problem can be found in the literature (see [15] for a list of references). The last to date, announced by Ramis, bears a striking resemblance to Abhyankar's conjecture on the coverings of algebraic curves in finite characteristic. As in the work of Kovacic [13], the author here searches for an algebraic (more specifically, a constructive) solution (unfortunately, Theorem 7.13 is incorrect as stated; see [15]). Let me here take advantage of this review to mention yet another algebraic version of these inverse problems, which does not seem to have attracted attention until now. As we saw in §3, the differential Galois groups G_K and G'_K one can attach to $(*)$ could be different over K. Thus, given a reductive K-algebraic group H' in $\mathrm{GL}_n(K)$, can one always construct a differential system $(*)$ over K whose *intrinsic* Galois group G'_K is K-isomorphic to H'?

There are many other aspects of the book, and in particular, a large supply of classical examples which illustrate the theory in a convincing way. One may regret that no mention is made of the more recent computations of differential Galois groups obtained for families such as Airy or hypergeometric equations (cf. [10], [3]) or for equations of low order (cf. [18]). But it is probably the price to pay for its success that it is not possible anymore to do justice to the development of differential Galois theory within the format of Kaplansky's book. To paraphrase [8], differential algebra is no longer "(99 per cent or more) the work of Ritt and Kolchin"; the self-contained introduction Magid's 100-page book provides should help the newcomer to proceed further into this beautiful and active field.

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