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Lie algebras with triangular decompositions, by Robert V. Moody and Arturo Pianzola, Canad. Math. Soc. Ser. Monographs Adv. Texts, Wiley-Interscience, New York, 1995, xx + 685 pp., ISBN 0-471-63304-6

The authors start from the viewpoint that finite-dimensional semisimple Lie algebras (over **C**, say) and infinite-dimensional generalizations that have been studied since about 1967 can and should be treated together. The title of the book reflects only a part of the abstract structure that is shared. The algebras of the title have a finite-dimensional commutative subalgebra $\mathfrak{h}$ whose action by multiplication on the full algebra is diagonalizable in such a way that the algebra $\mathfrak{g}$ has a direct decomposition

$$\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+,$$

where $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are "dual" subalgebras, normalized by $\mathfrak{h}$. One may take as a model $\mathfrak{g} = \mathfrak{gl}(m)$, the full linear Lie algebra, with $\mathfrak{h}$ the diagonal subalgebra and $\mathfrak{n}_+(\mathfrak{n}_-)$ the strictly upper (lower) triangular matrices. Under additional conditions, the linear functions on $\mathfrak{h}$ corresponding to the immediately superdiagonal matrix units (in the case of $\mathfrak{gl}(m)$) form a base for a "system of roots", and this phenomenon generalizes.

It has been known for more than a hundred years that, in the finite-dimensional case, an associated integral matrix (the "Cartan matrix") determines the algebra (W. Killing and E. Cartan), but it was only in the 1960s that Serre [1] showed that one has a presentation (generators and relations) for the Lie algebra using only the matrix.

Kac-Moody algebras, which generalize the theorem of Serre by generalizing the matrix of integers, form the ultimate object of study in the book. The authors give a more general preparation of the ground in the first three chapters of roughly a hundred pages each. After a quite gentle first chapter on definitions and examples, they proceed in the second chapter to algebras in the generality of the title. Here it is possible to develop analogues of important features of the representation theory for finite-dimensional algebras (as found, for example, in the book of Humphreys [2]), treating a "category $\mathcal{O}$", inclusions of "Verma modules", etc., and only stopping short of Casimir operators and Weyl's character formula. These last have to wait for the definition by (symmetrizable generalized Cartan) matrices in Chapter 4 and a detailed study of the modules for Kac-Moody algebras in Chapter 6.

A rather dramatic, if simplistic, demonstration of the patience of the authors may be obtained by comparing page 330 of the book with the first page of that by Kac. In each case, this is the site of introduction of generalized Cartan matrices and Lie algebras defined by them. Earlier chapters enable an efficient exploitation of the power given by the new conditions. In particular, one now has an analogue, due to Kac, of the Casimir operator and the structure theorem of Gabber and Kac [3], reconciling the independent original constructions of Kac [4] and of Moody [5] and clarifying the analogy with Serre's result.

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The fifth chapter of nearly ninety pages treats systems of roots of the kind associated with Kac-Moody algebras axiomatically in their own right, with a Weyl group but with "dual roots" having to be prescribed rather than being derived from a scalar product. This chapter, which could be read as a self-contained (if not particularly well-motivated) topic by itself, lays the groundwork for a more detailed study of "integrable modules" in Chapter 6, where one is able, because of local nilpotency of many elements of the Lie algebra, to form exponentials in finite terms. A group G is then associated with the Lie algebra, G having the property of being generated by generators formally corresponding to a suitable family of elements of the Lie algebra, with G represented on each integrable module by the group generated by the images of the exponentials of these elements. A somewhat larger group $\tilde{G}$ accommodates additional elements that act diagonally on integrable modules. The actions of these groups are studied, and proofs are given of results of Kac-Peterson [6], Kac-Kazhdan [7], Bernstein-Gelfand-Gelfand [8], Jantzen (translation functors), and Kac-Wakimoto [9], extending most of the remainder of the representation theory for finite-dimensional algebras. (A survey of constructions of groups associated with Kac-Moody algebras may be found in Tits [10].)

The final chapter (Chapter 7) carries over additional representation-theoretical and structural results of a more technical nature.

Much of the interest in the generalization to Kac-Moody algebras has been stimulated by discovery of surprising connections with classical mathematics and with their employment as models in modern physics. The book by Kac brings out a number of the mathematical connections in its later chapters. One of these, a connection via character formulas with partitions and theta functions, is treated in an appendix to the work at hand. In this appendix, the algebra $A_1^{(1)}$ associated with the generalized Cartan matrix

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

is studied in detail, recapitulating chapter by chapter what the general results mean in its case.

The style of the book is that of an advanced textbook, along the lines of various Bourbaki *Éléments*, rather than that of a research monograph. Thus a reader already moderately familiar with abstract algebra may be surprised at the inclusion of some quite elementary material. Following the good example of Bourbaki, the authors have capped each chapter with a substantial set of exercises, often developing important results beyond those in the text. In prefaces "How to Read This Book" and "Course Outlines", the authors confirm their view of it as a textbook by illustrating how one might organize sections of the book for study with specific objectives.

As is often the case with textbooks, the authors offer little overview or motivational matter at the beginning. In this respect, they suffer by comparison with Kac [11], where the introduction is a model to be admired and emulated. It is of course possible to argue that many of those for whom Moody and Pianzola intend their book would lack the sophistication that Kac presupposes. Still, the advice of the reviewer to a novice would be to read Kac's introduction before delving far into this book and to reread it periodically to have a sense for the bigger picture.

In a work of nearly 700 pages, it would be surprising if there were no mathematical errors. That my reading did not discover any is probably testimony as much to the superficiality of that reading as to the care exercised by the authors. On the other hand, about a hundred misprints and similar instances of carelessness, some twenty-five in the bibliography, did attract notice. Many of these were in French or German titles and unlikely to cause difficulties. However, they hardly inspire confidence if one is browsing in the book. The authors issue a disclaimer concerning the bibliography that only items that are referred to in the text have been included. They further excuse omissions by referring to four other sources, one of which, [LMS], is not included in their bibliography nor given any further identification. Thus no available list of references is newer than 1990, and the bibliography contains only three references newer than 1991, two of them "to appear", with one of the authors as joint author. In other words, although the introduction is dated 1994, we are left with no bibliography that is less than five years old, and this in a field of vibrant activity during recent years. Another missed opportunity.

Reference in the bibliography to the book by Kac indicates the second (1985) edition, not much updated from the first (1983). There is no indication as to the existence of a third (1990) edition [11], much enlarged and brought up to date, particularly in aspects related to physics. Where Moody and Pianzola treat developments of the years since Kac's second edition, they are able to lay the groundwork with considerable aforethought. They approach their task with a commitment to thoroughness and expository care that will be appreciated by the patient reader. That reader will have the delight of the leisurely development of a rich and beautiful structure.

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