BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 3, July 1996

The harmonic analysis of probability measures on hypergroups, by Walter R. Bloom and Herbert Heyer, de Gruyter Stud. Math., vol. 20, de Gruyter, Berlin and Hawthorne, NY, 1995, vi + 601 pp., $125.00, ISBN 3-11-012105-0

The natural abode for harmonic analysis is a locally compact topological group G and the various subspaces of the group measure algebra, denoted $(M(G), *)$, obtained by endowing $M(G)$, the space of finite Borel measures on G, with the convolution $*$ defined by $(\mu * \nu)(E) = \int_G \mu(Ex^{-1})\, d\nu(x)$. Consequently those parts of probability theory that exploit ideas like convolution of measures and characteristic functions (i.e., Fourier transforms of probability distributions) are most successfully pursued in the context of a group, for if X and Y are independent random variables in G with probability distributions μ and ν, then $X \cdot Y$ has distribution $\mu * \nu$. This is the subject of an earlier volume [Hey77] by Professor Heyer.

There has also been much work in harmonic analysis and probability theory in settings where no group is present, but where group-like methods are still applicable; we will begin by giving three families of examples. In each case, there is an underlying space H which will be one of $I = [-1, 1]$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, or $\mathbb{R}_+ = [0, \infty)$. In each case, the space of bounded Borel measures on H becomes an algebra with a product (called convolution) that does not depend on the arithmetic of the underlying space H. In fact, for each choice of H there is a continuum of distinct convolutions. The resulting convolution measure algebras will each have sufficient resemblance to the convolution measure algebra of a locally compact group to make it an interesting and useful setting for studies in harmonic analysis and probability, yet they are all strikingly different from the measure algebra of a locally compact group. For instance, the convolution of two point masses will not be another point mass as in the group case; it may, in fact, even be absolutely continuous with respect to Lebesgue measure.

As usual, $C(H)$ denotes the continuous functions on H, $M(H)$ the bounded Borel measures on H, and $M_1(H)$ the probability measures on H. The support of a measure μ is denoted $\operatorname{supp}(\mu)$, and the unit point mass at x will be denoted δ_x.

Example 1. Orthogonal polynomials. I. I. Hirschman, Jr., pointed out in 1956 [Hir56b, Hir56a] that the structure for harmonic analysis exists in a setting where certain orthogonal polynomials could play the role of the exponentials in classical Fourier analysis. For example, consider the Legendre polynomials $\{P_n\}_{n\in\mathbb{N}_0}$. These are orthogonal with respect to Lebesgue measure on I and are normalized by requiring $P_n(1) = 1$. The Legendre polynomials satisfy a product formula:

$$P_n(x)P_n(y) = \int_I K(x, y, z)P_n(z)dz \qquad (-1 < x,\, y < 1)$$

1991 *Mathematics Subject Classification.* Primary 43A62, 60B99; Secondary 43A10, 28A33, 43A35.

This review was prepared during the tenure of National Science Foundation grant DMS-9404316.

with

$$K(x,y,z) \geq 0 \quad \text{and} \quad \int_I K(x,y,z)dz = 1.$$

For $f,\, g \in L^1 = L^1(I, dx)$ define

$$(f * g)(z) = \int_I \int_I K(x,y,z) f(x) g(y)\, dx\, dy$$

so that

$$\int_I (f*g)(x) P_n(x)\, dx = \left[\int_I f(x) P_n(x)\, dx\right] \cdot \left[\int_I g(x) P_n(x)\, dx\right],$$

and it follows that $(L^1, *)$ is a Banach algebra. The operation is easily extended to the point masses by defining:

$$\begin{aligned} &d(\delta_x * \delta_y)(z) = K(x,y,z)\, dz, && (-1 < x,\, y < 1), \\ &\delta_x * \delta_1 = \delta_x \quad \text{and} \quad \delta_x * \delta_{-1} = \delta_{-x}, && (x \in I). \end{aligned}$$

Finally if $\mu,\, \nu \in M(I)$, define $\mu * \nu$ by its action on an arbitrary continuous function:

$$\int_I f d(\mu * \nu) = \int_I \int_I \int_I f\, d(\delta_x * \delta_y)\, d\mu(x)\, d\nu(y), \qquad (f \in C(H)).$$

There is also a natural dual operation on $\ell^1 = M(\mathbb{N}_0)$ based on the linearization formula:

$$P_n(x) P_m(x) = \sum_{k=0}^{n+m} c_{n,m}^k P_k(x).$$

The coefficients satisfy

$$c_{n,m}^k \geq 0 \quad \text{and} \quad \sum_{k \in \mathbb{N}_0} c_{n,m}^k = 1.$$

Thus if $a,\, b \in \ell^1$, their convolution $a \star b$ is defined by

$$(a \star b)_k = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m}^k a_n b_m$$

so that for each $x \in I$,

$$\sum_{k \in \mathbb{N}_0} (a \star b)_k P_k(x) = \left[\sum_{n \in \mathbb{N}_0} a_n P_n(x)\right] \cdot \left[\sum_{m \in \mathbb{N}_0} b_m P_m(x)\right].$$

Thus Hirschman describes two measure algebras: $(M(I), *)$ and $(M(\mathbb{N}_0), \star)$. These have many properties analogous to the classical convolution algebras of measures on the circle group and its dual on the group of integers; for instance, a convolution of probability measures is always a probability measure.

Actually, Hirschman discusses these constructions not just for the Legendre polynomials, but for the ultraspherical polynomials $P_n^{(\lambda)}(x)$ which are orthogonal on I with respect to the measure $(1-x^2)^{\lambda - \frac{1}{2}}\, dx$. The normalized polynomials $R_n^{(\lambda)}(x) = P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1)$ are used in place of the Legendre polynomial $P_n(x)$. So for each $\lambda \geq -1/2$ Hirschman obtains a pair of dual measure algebras $(M(I), *_\lambda)$ and $(M(\mathbb{N}_0), \star_\lambda)$. It is important to note $*_\lambda$ is a distinct convolution for each $\lambda \geq -1/2$; hence a continuum of Banach *algebras* are built on the single Banach *space* $M(I)$. The algebraic structure does not depend on any arithmetic in the underlying space I. Similar remarks hold for $(M(\mathbb{N}_0), \star_\lambda)$.

In the special case when $n = 2\lambda + 2$ is an integer, these structures can also be inherited from a group, since $(M(I), *_\lambda)$ is isometrically isomorphic to the subalgebra of $M(SO(n))$ consisting of measures which are bi-invariant with respect to the action of $SO(n-1)$, or equivalently the measures on the unit sphere in $\mathbb{R}^n$ which are invariant under all rotations that leave some designated point fixed. Thus $(M(I), *_\lambda)$ and $(M(\mathbb{N}_0, \star_\lambda)$ interpolate these in some sense.

For most values of λ the group structure is absent, yet there is still an adequate structure in $(M(I), *_\lambda)$ to define and study objects like Fourier multipliers, maximal functions, and a Littlewood-Paley theory (see [CS77] and the references cited there).

Example 2. Spherically symmetric random walks. This example is the subject of a 1963 article by J. F. C. Kingman [Kin63]. Consider a pair of independent random variables $\mathbf{X}$ and $\mathbf{Y}$ in $\mathbb{R}^2$ with lengths X and Y, but with direction uniformly distributed. The sum $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$ also has uniformly distributed direction, but its length $Z = |\mathbf{Z}|$ is a random number in the range $|X - Y| \leq Z \leq X + Y$. In general, if $\mathbf{X}$ and $\mathbf{Y}$ are independent random variables in $\mathbb{R}^2$ with uniformly distributed direction, but with lengths X and Y having probability distributions μ, $\nu \in M_1(\mathbb{R}_+)$, then Z is a random variable in $\mathbb{R}_+$ with a probability distribution depending on μ and ν which is denoted by $\mu \circ \nu$, and we write $Z = X \oplus Y$. The operation $\circ$ is readily extended to all of $M(\mathbb{R}_+)$ so that $(M(\mathbb{R}_+), \circ)$ becomes a measure algebra which is isometrically isomorphic to the subalgebra of the group convolution algebra $(M(\mathbb{R}^2), *)$ consisting of the measures invariant with respect to rotations of the plane. The role played by the Legendre polynomials in the first example is played here by the Bessel functions $J_0(xy)$ in the sense that they satisfy a product formula which yields

$$\int_{\mathbb{R}_+} J_0(xy)\, d(\mu * \nu)(x) = \left[\int_{\mathbb{R}_+} J_0(xy)\, d\mu(x)\right] \cdot \left[\int_{\mathbb{R}_+} J_0(xy)\, d\nu(x)\right]$$

so that the useful substitute for the characteristic function of the random variable X is

$$\Phi_X(y) = \int_{\mathbb{R}_+} J_0(xy)\, d\mu(x).$$

The product formula for the Bessel functions also ensures the fundamental property of characteristic functions

$$\Phi_{X \oplus Y} = \Phi_X \Phi_Y$$

when X and Y are independent random variables in $\mathbb{R}_+$.

Kingman actually describes a continuum of measure algebras $(M(\mathbb{R}_+), \circ_\alpha)$ related to Bessel functions of order $\alpha \geq -1/2$. These correspond to the subalgebras of rotation invariant measures on $\mathbb{R}^n$ when $n = 2\alpha + 2$ is an integer. In the other cases, there is again no useful algebraic structure in the underlying spaces. Nevertheless Kingman is able to define random walk and Brownian motion and obtain a law of large numbers, a central limit theorem, a recurrence theorem, and characterizations of infinitely divisible and stable distributions. When $n = 2\alpha + 2$ is an integer, all of this is an inheritance from the group structure on $\mathbb{R}^n$, but Kingman obtains his results for all real $\alpha \geq -1/2$ with no reference to the group case except for inspiration.

Hypergroups. Each of $(M(I), *_\lambda)$, $(M(\mathbb{N}_0, \star_\lambda)$ and $(M(\mathbb{R}_+), \circ_\alpha)$ is a hypergroup. Roughly speaking, a hypergroup is a measure algebra which has many of the useful properties associated with the convolution measure algebra of a group, but no algebraic structure is presumed for the underlying space. For the sake of clarity we will first give the hypergroup axioms as they apply to the case where the underlying space H is discrete; this will be followed by the modifications for the compact and locally compact cases. Thus we assume H is discrete and that $M(H)$ is a Banach algebra with product $*$. Then $(H, *)$ is a *hypergroup* (or *hypergroup measure algebra*) if it has the following properties:

(H1) If μ and ν are probability measures on H, then so is $\mu * \nu$.
(H2) There is an element $e \in H$ such that $\delta_e * \mu = \mu * \delta_e$ for all $\mu \in M(H)$.
(H3) There is a continuous involution $x \mapsto x^\vee$ $(x^{\vee\vee} = x)$ such that $e \in \operatorname{supp} \delta_x * \delta_y$ if and only if $y = x^\vee$.
(H4) $(\mu * \nu)^\vee = \nu^\vee * \mu^\vee$ where $\mu^\vee$ is defined by $\int_H f(x)\, d\mu^\vee(x) = \int_H f(x^\vee)\, d\mu(x)$.
(H5) $\operatorname{supp} \delta_x * \delta_y$ is finite.

If we assume H is compact, then (H5) must be modified and we must add a sixth axiom:

(H5) $(x, y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ is continuous with an appropriate topology for the space of compact subsets of H.
(H6) $(\mu, \nu) \mapsto \mu * \nu$ is weak-* continuous.

Finally, in the locally compact case, we must add to (H5) the requirement that $\operatorname{supp}(\delta_x * \delta_y)$ is compact, and in (H6) "weak-* continuous" must be replaced by another form of weak continuity.

The definitive set of axioms was given first by Jewett in his encyclopaedic article [Jew75]. Jewett calls these objects "convos" and, ironically, never once uses the term "hypergroup". The interested reader is directed to Jewett [Jew75] for a precise statement of the axioms.

We also need the following two definitions: m is a *Haar measure* for the hypergroup $(H, *)$ if for every $x \in H$, $m * \delta_x = \delta_x * m = m$. $\phi \in C(H)$ is a *character* of $(H, *)$ if ϕ is bounded, $\phi(x^\vee) = \overline{\phi(x)}$ and

$$\int_H \phi\, d(\delta_x * \delta_y) = \phi(x)\phi(y) \qquad (x,\, y \in H).$$

If $*$ is commutative, the set of characters with the weak topology defines an object which is, in some sense, dual to $(H, *)$.

The measure algebra of a group with identity e is an obvious example of a hypergroup; convolution is defined by $\delta_x * \delta_y = \delta_{xy}$ and $x^\vee = x^{-1}$, the group inverse of x. In the first example, $(I, *_\lambda)$ is a hypergroup with $e = 1$ and $x^\vee = x$, while $(\mathbb{N}_0, \star_\lambda)$ is a hypergroup with $e = 0$ and $x^\vee = x$. In the second example, $(\mathbb{R}_+, *)$ is a hypergroup with $e = 0$ and $x^\vee = x$.

Jewett [Jew75] obtains most of the basic properties necessary to do analysis in hypergroups. In an earlier article Dunkl [Dun73] had shown that a hypergroup $(H, *)$ has a Haar measure if H is compact and $*$ is commutative. Jewett [Jew75] shows there is a Haar measure when H is either compact or discrete. Spector [Spe78] showed $(H, *)$ has a left-Haar measure when $*$ is commutative. Both Dunkl and Spector use definitions for hypergroup that are slightly different from Jewett's, but their results still apply to Jewett's convos. This is one of the reasons that the object we call "hypergroup" is now often referred to as a "DJS hypergroup".

There are striking contrasts between hypergroups and group measure algebras, for instance:

- In a group, a convolution of point masses is another point mass, but this is not the case in a hypergroup. Indeed, if the point masses of a hypergroup constitute a semigroup, then the hypergroup is the measure algebra of a group.
- Every group has a Haar measure, but this question is still open for general hypergroups.
- Every locally compact abelian group G has a dual locally compact abelian group Γ and the dual of Γ is again G. The analogous fact is true for the examples given above: (I, λ_*) and $(\mathbb{N}_0, \star_\lambda)$ are duals of one another and $(\mathbb{R}_+, \circ_\alpha)$ is self-dual, but there are examples based on Gasper's product and linearization formulas for Jacobi polynomials [Gas70, Gas72] for which duality fails. There are several possibilities for a commutative hypergroup $(H, *)$: the dual object of $(H, *)$ may or may not be a hypergroup; if it is, the second dual may or may not be $(H, *)$.
- The dual object of a commutative hypergroup will have a Plancherel measure, but its support may or may not be the entire dual; the support might not even include the trivial character which is identically one.

One reason for the rapid growth of the field is that there is an enormous diversity of interesting and useful examples, for instance the three hypergroups $(I, *_\lambda)$, $(\mathbb{N}_0, \star_\lambda)$, $(\mathbb{R}_+, \circ_\alpha)$ can be generalized in several ways:

- The ultraspherical polynomials can be replaced by other families of orthogonal polynomials to obtain hypergroups analogous to (I, λ_*) and $(\mathbb{N}_0, \star_\lambda)$; we call these *continuous* and *discrete polynomial hypergroups*. The entire category of continuous polynomial hypergroups has been identified [CS90], but the class of discrete polynomial hypergroups is known to be much larger [Szw92a, Szw92b].
- There are continuous and discrete polynomial hypergroups in the multivariate case also, but the analysis and understanding of these categories is in a very early stage [CS95, KS95].
- The characters of $(I, *_\lambda)$ and $(\mathbb{R}_+, \circ_\alpha)$ are eigenfunctions of Sturm-Liouville problems. There are theorems giving sufficient conditions on a Sturm-Liouville problem that its eigenfunctions be characters of a hypergroup; these are called *Sturm-Liouville hypergroups.*
- Any product formula for a family of functions such as the ones for ultraspherical polynomials and Bessel functions can be used to define, at least in some formal sense, a measure algebra which may, in fact, be a hypergroup [Mar89, CMS92, CMS93].

Interest in hypergroups has grown for several reasons:

- A hypergroup is a natural setting for probabilistic problems based on the following notion of addition of random variables: if $(H, *)$ is a hypergroup and X and Y are independent random variables in H with distributions μ and ν, then $X + Y$ is defined to be a random variable with distribution $\mu * \nu$. One can go on to formulate the same definitions and problems as in the Kingman example.
- In harmonic analysis and probability theory there is often interest in measures on a group which are invariant under certain symmetries; this is the case for the Hirschman and Kingman examples for special values of the parameters λ

and α. The corresponding family of measures on the orbit space is often a hypergroup.

- There had already existed a large body of work concerned with expansions of functions in bases of special functions or eigenfunctions of Sturm-Lioville problems that turn out to be characters of a hypergroup.
- There are challenging questions about general hypergroups such as the existence of Haar measure. Many ideas which make sense in classical Fourier analysis—for instance, Sidon sets, the F. and M. Riesz theorem, multipliers, maximal functions, etc.—have natural interpretations in general hypergroups, but many of the problems are challenging even in specific examples of hypergroups [Kan76, GS95].
- Workers in other similar systems, such as quantum groups and hypercomplex systems, have become interested in connections between these systems and hypergroups. (See the conference proceedings [CGS95].)

The book. The authors of the present volume have been collaborating on hypergroup research for about fifteen years; in addition, Heyer has nurtured a substantial German school of hypergroup research, and he has provided an international forum for the subject by including hypergroups as a significant topic in the series of conferences "Probability Measures on Groups" at Oberwolfach (for instance, see the conference proceedings [Hey84a, Hey86, Hey91, Hey89, Hey95]. Now Bloom and Heyer have produced a major book on the subject which will be welcomed by workers in the field as a working resource and as a record of progress.

About twenty years have elapsed since the articles of Dunkl, Jewett and Spector, and in that time the field has grown by attracting workers in special functions, probability, and harmonic analysis. Students of hypergroups relied mainly on Jewett's paper as a text (the lucky ones of us also have a copy of Kenneth Ross's index of that paper) and for more recent developments the surveys of Heyer [Hey84b] and Litvinov [Lit87] (which also treats developments in other systems similar to hypergroups). The field is young and still growing rapidly in many directions, and the book is necessarily a snapshot of its current state.

The volume has three obvious audiences: harmonic analysts, probabilists and workers in special functions. Harmonic analysts will be primarily interested in the first four chapters of the book, where there are treatments of invariant measures, idempotents, duality, positive and negative definiteness. Probabilists will turn to the latter half to find topics which include infinite divisibility, factorization, transience and renewal, strong laws of large numbers and central limit theorems. What is often very interesting here is the diverse interpretations of various results in different types of hypergroups. What might be a single theorem in the group case may take many forms in the different types of hypergroups. The authors do a good job of formulating the results in the most general settings, but they also give interpretations in the special cases. This has the effect of keeping the book interesting to a wider audience. For example, §5.4, which discusses factorization, is about evenly split between factorization in general hypergroups and factorization in two important classes of examples: hypergroups on $\mathbb{N}_0$ which arise from orthogonal polynomials as in Example 1 and certain hypergroups whose characters are eigenfunctions of second-order linear differential equations as in Example 2.

Workers in special functions, and indeed a much wider audience, will find the extensive treatment of examples useful and will value the book as a guide to a large body of literature. The book's 48-page bibliography covers not only works which are specifically about hypergroups, but a large body of earlier work which, though it makes no explicit mention of hypergroups, belongs under this rubric. The chapter-end notes and the bibliography will quickly lead the interested reader to the background and current state of knowledge on most of the topics in the book. A full treatment of the concrete examples of hypergroups would probably fill a book at least as big as the one under review. Nevertheless, this volume contains a generous treatment of many examples. Chapter 1 discusses various hypergroups which arise from general groups; Chapter 3 is devoted almost entirely to discrete polynomial hypergroups (the authors call them simply "polynomial hypergroups"), hypergroups where H is a one-dimensional set, and Sturm-Liouville hypergroups of $\mathbb{R}_+$. This chapter includes dozens of examples, and additional ones are scattered through the text. A list at the end of the book contains about 80 examples (many of which are themselves parametric families), and even this list is not nearly exhaustive.

Of course, no work of this size can reach the press without a few glitches; most of those noted by the reviewer are obvious misprints, but we note that the authors have overlooked the need for the requirement of some kind of weak continuity for the convolution in the definition of a hypergroup on page 9. This is required in order to be able to reconstruct the entire measure algebra from the convolution of point masses by means of a formula on page 10. This small problem does not detract from the value of the book; anybody who wishes to work in this field will need to have a copy near at hand.

References

[CGS95] W. C. Connett, O. Gebuhrer, and A. L. Schwartz (eds.), *Applications of hypergroups and related measure algebras*, American Mathematical Society, Providence, R. I., 1995, Contemporary Mathematics, **183**. MR **95m:**43001

[CMS92] W. C. Connett, C. Markett, and A. L. Schwartz, *Convolution and hypergroup structures associated with a class of Sturm-Liouville systems*, Trans. Amer. Math. Soc. **332** (1992), 365–390. MR **92j:**43005

[CMS93] ———, *Product formulas and convolutions for angular and radial spheroidal wave functions*, Trans. Amer. Math. Soc. **338** (1993), 695–710. MR **93j:**33011

[CS77] W. C. Connett and A. L. Schwartz, *The theory of ultraspherical multipliers*, Mem. Amer. Math. Soc. **183** (1977), 1–92. MR **55:**8666

[CS90] ———, *Product formulas, hypergroups, and Jacobi polynomials*, Bull. Amer. Math. Soc. (N.S.) **22** (1990), 91–96. MR **90h:**33022

[CS95] ———, *Continuous 2-variable polynomial hypergroups*, Applications of hypergroups and related measure algebras (O. Gebuhrer W. C. Connett and A. L. Schwartz, eds.), American Mathematical Society, Providence, R. I., 1995, Contemporary Mathematics, **183**, pp. 89–109. MR **1:**334 773

[Dun73] C. F. Dunkl, *The measure algebra of a locally compact hypergroup*, Trans. Amer. Math. Soc. **179** (1973), 331–348. MR **47:**9171

[Gas70] G. Gasper, *Linearization of the product of Jacobi polynomials. II*, Canad. J. Math. **32** (1970), 582–593. MR **41:**8732

[Gas72] ———, *Banach algebras for Jacobi series and positivity of a kernel*, Ann. of Math. **95** (1972), 261–280. MR **46:**9634

[GS95] O. Gebuhrer and A. L. Schwartz, *Sidon sets and Riesz sets for some measure algebras on the disk*, to appear, 1995.

[Hey77] H. Heyer, *Probability measures on locally compact groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 94, Springer-Verlag, Berlin, 1977. MR **58:**18648

[Hey84a] ———, (ed.), *Probability measures on groups VII, Proceedings Oberwolfach 1983*, Lecture Notes in Math., vol. 1064, Springer, Berlin, 1984. MR **85h:**60002

[Hey84b] ———, *Probability theory on hypergroups: a survey*, Probability Measures on Groups VII, Proceedings Oberwolfach 1983 (Berlin) (H. Heyer, ed.), Springer, 1984, Lecture Notes in Math., vol. 1064, pp. 481–550. MR **86c:**60016

[Hey86] ———, (ed.), *Probability measures on groups VIII, Proceedings Oberwolfach 1985*, Lecture Notes in Math., vol. 1210, Springer, Berlin, 1986. MR **87j:**60006

[Hey89] ———, (ed.), *Probability measures on groups IX, Proceedings Oberwolfach 1988*, Lecture Notes in Math., vol. 1379, Springer, Berlin, 1989. MR **90i:**60001

[Hey91] ———, (ed.), *Probability measures on groups X, Proceedings Oberwolfach 1990*, Plenum, New York, 1991. MR **93f:**60001

[Hey95] ———, (ed.), *Probability measures on groups and related structures XI, Proceedings Oberwolfach 1994*, World Scientific, Singapore, 1995.

[Hir56a] I. I. Hirschman, Jr., *Harmonic analysis and the ultraspherical polynomials*, Symposium of the Conference on Harmonic Analysis (Cornell), 1956.

[Hir56b] ———, *Sur les polynomes ultraspheriques*, C. R. Acad. Sci. Paris **242** (1956), 2212–2214. MR **17:**1113d

[Jew75] R. I. Jewett, *Spaces with an abstract convolution of measures*, Adv. in Math. **18** (1975), 1–101. MR **52:**14840

[Kan76] Y. Kanjin, *A convolution measure algebra on the unit disc*, Tôhoku Math. J. (2) **28** (1976), 105–115. MR **53:**1178

[Kin63] J. F. C. Kingman, *Random walks with spherical symmetry*, Acta Math. **109** (1963), 11–53. MR **26:**7052

[KS95] T. H. Koornwinder and A. L. Schwartz, *Product formulas and associated hypergroups for orthogonal polynomials on the simplex and on a parabolic biangle*, preprint, 1995.

[Lit87] G. L. Litvinov, *Hypergroups and hypergroup algebras*, J. Soviet Math. **38** (1987), 1734–1761. MR **87j:**43002

[Mar89] C. Markett, *Product formulas and convolution structure for Fourier-Bessel series*, Constr. Approx. **5** (1989), 383–404. MR **90i:**33012

[Spe78] R. Spector, *Mesures invariantes sur les hypergroupes*, Trans. Amer. Math. Soc. **239** (1978), 147–165. MR **57:**3745

[Szw92a] R. Szwarc, *Orthogonal polynomials and a discrete boundary value problem I*, SIAM J. Math. Anal. **23** (1992), 959–964. MR **93i:**33007

[Szw92b] ———, *Orthogonal polynomials and a discrete boundary value problem II*, SIAM J. Math. Anal. **23** (1992), 965–969. MR **93i:**33007

Alan L. Schwartz
University of Missouri-St. Louis
E-mail address: `schwartz@arch.umsl.edu`