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Lattice methods for multiple integration, by Ian H. Sloan and Stephen Joe, Oxford University Press, New York and Oxford, 1994, xi+239 pp., $69.95, ISBN 0-19-853472-8

The book under review discusses a class of methods, called *lattice methods*, for the approximation of multiple integrals

$$If = \int_{C^s} f(\vec{x})d\vec{x} = \int_0^1 \dots \int_0^1 f(x_1, \dots, x_s)dx_1 \cdots dx_s. \tag{1}$$

Lattice rules are designed to approximate such integrals in situations in which f is one-periodic with respect to each component of $\vec{x}$ and in addition is reasonably smooth. This book does not discuss only the theoretical aspects of lattice rules. The practical aspects are not avoided, and at the end the implementation of an integration algorithm is studied.

A basic concept this book builds upon is the so-called *integration lattice*, introduced in Chapter 2. This is a discrete subset of $\mathbb{R}^s$ which is closed under addition and subtraction and which contains $\mathbb{Z}^s$ as a subset. A lattice rule for the approximation of If (1) is the average value of the integrand over all points of an integration lattice L inside the integration region C^s, i.e.

$$Qf = \frac{1}{N}\sum_{j=0}^{N-1} f(x_j), \quad \{x_j : j = 0, \dots, N-1\} = L \cap C^s.$$

An example of a lattice rule for 1-dimensional integration is the rectangle rule,

$$R_n f = \frac{1}{n}\sum_{j=0}^{n-1} f\left(\frac{j}{n}\right).$$

How can one generalise the rectangle rule to higher dimensions? This question does not have a unique answer. The most obvious generalisation is the product-rectangle rule, but this rule is not cost-effective if the dimension s is high. The rule requires $N = n^s$ integrand evaluations, and the rate of convergence can be very slow when expressed in terms of N. The oldest interesting generalisation of the rectangle rule is the so-called *method of good lattice points* which uses approximations of the form

$$Qf = \frac{1}{N}\sum_{j=0}^{N-1} f\left(\frac{j}{N}\vec{z}\right)$$

where N is an a priori chosen number of cubature points and $\vec{z}$ a carefully selected integer vector. The resulting rules are often called *number-theoretic rules*. This is not surprising if one knows the number-theoretic properties of N and $\vec{z}$ turn out to be significant for the accuracy of this type of integration rule.

So, the time has come to consider the accuracy of a lattice rule. As for the rectangle rule, the error of an arbitrary lattice rule can be expressed in terms of the

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Fourier components of f, provided the periodic extension of f has an absolutely convergent Fourier series

$$f(\vec{x}) = \sum_{h \in \mathbb{Z}^s} \hat{f}(\vec{h}) e^{2\pi i \vec{h} \cdot \vec{x}}.$$

With every lattice L one can associate a *dual lattice* L^T, namely the set

$$\{\vec{h} \in \mathbb{R}^s : \vec{h} \cdot \vec{x} \in \mathbb{Z} \text{ forall } \vec{x} \in L\}.$$

This dual lattice plays an important role in the study of the error of a lattice rule. It is proven that this error satisfies

$$Qf - If = \sum_{h \in L^T}{}' \tilde{f}(h).$$

In order to study lattice rules in a systematic way, a classification scheme that recognises different types of lattices is useful. This is the subject of Chapter 3. Several classification schemes for lattice rules exist and are treated in this book. The most important one is probably the one based on the concept of *rank*. The rank of a lattice rule is the minimum number of (embedded) sums required to write it. For example, the method of good lattice rules deals with rank-1 rules; the product rectangle rule in s dimensions is a rank-s rule.

The study of the different forms in which lattice rules can be written is a nice application of group theory. The book is self-contained, and so it contains the necessary background on this topic.

In Chapter 4, the book studies rank-1 rules in detail. A good rank-1 rule is based on a good choice of $\vec{z}$. Obviously, many criteria are possible for measuring "goodness". All are related to number theory. One criterion is based on a criterion $P_\alpha(\vec{z}, N)$, which provides an upper bound for the quadrature error for test functions belonging to a certain smoothness class. Another criterion is based on $p(\vec{z}, N)$, the *Zaremba index* or *figure of merit*:

$$p(\vec{z}, N) := \min_{\substack{\vec{h} \cdot \vec{z} = 0 \pmod N \\ \vec{h} \neq 0}} \bar{h}_1 \bar{h}_2 \cdots \bar{h}_s \quad \text{with} \quad \bar{h}_i = \max(1, |h|).$$

A good rule is one in which $p(\vec{z}, N)$ is as large as possible. If the integrand f is of a lesser smoothness, e.g. does not have a periodic extension or an absolutely convergent Fourier series, then the *discrepancy* of the integration points provides a criterion.

In two dimensions the so-called Fibonacci rules achieve the best rate of convergence. These are rank-1 rules of the form

$$Q_k f = \frac{1}{F_k} \sum_{j=0}^{F_{k-1}} f\left(\left\{\frac{1}{F_k}(1, F_{k-1})\right\}\right)$$

where F_k is the k-th Fibonacci number and $\{x\}$ is the fractional part of x.

In higher dimensions no explicit construction of good rules is known. The theory provides us with theorems that guarantee the existence of good sequences $\vec{z}(N)$ for which $P_\alpha(\vec{z}, N)$ converges to 0 as N goes to infinity. The main tool to construct rules is by means of a search method.

After the study of rank-1 rules, the book proceeds in Chapters 5, 6 and 7 with the study of rules of a higher rank. The measures for goodness are generalised for general lattice rules. It is shown that lattice rules of general rank can achieve the

same order of convergence as rank-1 rules. However, the order of convergence is not decisive. The constant factors before the power of N are also important. The first evidence that higher rank rules are competitive with rank-1 rules came from numerical experiments. Since then, the relation between rules of rank 1 and rules of a higher rank has also been the subject of theoretical studies. There are now some grounds to believe lattice rules of maximal rank have advantages.

So far, it has been assumed that the integrand f was a one-periodic function. One can force periodisation by nonlinear transformations, e.g. polynomial transformations or trigonometric transformations. Obviously, this causes an overhead. Much worse however is the fact that such transformations tend to transform nice integrands into nasty ones. Chapter 8 studies an alternative: the lattice rules are slightly modified and then applied to the non-periodic integrand. One starts from a rank-1 rule with just one point, the origin, at the boundary of the region of integration. Then the contribution of the origin is replaced by a weighted sum of the contributions of all vertices of the unit cube. Obviously, for periodic integrands this does not make a difference. The question is then what the optimal choice for these weights is. The theoretical work on this is still in its infancy. Many questions are yet unanswered, and it is still not clear whether this approach is a valid alternative compared to the periodising transformations studied earlier in the book.

Chapter 9 is devoted to some theoretical questions about lattice rules that did not fit very well in the other chapters. First, it is investigated how many lattice rules with N points exist. With the current state of the art, geometrically equivalent lattices are counted twice. Two lattices are geometrically equivalent if one can be obtained from the other by applying the symmetries of the cube. It is also investigated how lattice rules can be used for integration over the space $\mathbb{R}^s$.

Chapter 10 is devoted to the practical implementation of lattice rules. A sequence of embedded lattice rules, which contains rules of ranks 1 to s, is studied. Such embedded sequences of integration rules are useful to derive error estimates with reasonable extra cost. The more precise rules in the sequence re-use all integrand evaluations performed by the less precise rules. The difference between two integration rules provides an error estimate for the less precise one. If more rules are available, one can make more sophisticated combinations to obtain more effective error estimates.

In the last chapter of the book numerical tests are presented that compare an algorithm based on an embedded sequence of lattice rules, an algorithm based on rank-1 rules, an adaptive Monte Carlo method and a global adaptive algorithm based on rules exact for polynomials. For the last three algorithms an implementation of the NAG library was used. The test used is a performance profile test for six families of integrands, each with its own type of difficulty. The conclusion of these tests is that the new algorithm is a very useful tool worthwhile to be added to the tool box of everyone who encounters multivariate integrals. It is available via netlib in the toms library. See also [1, 2].

At the end of the book, tables of good vectors $\vec{z}$ are provided along with a very complete list of references. (Reference [2] could not be included.)

I can recommend this book to everyone who wants or needs to know more about computing multivariate integrals. This book gives a very good overview of the current state-of-the-art of lattice rules, and it is very readable.

References

[1] S. Joe and I.H. Sloan, *Implementation of a lattice method for numerical multiple integration*, ACM Trans. Math. Software **19** (1993), no. 4, 523–545.

[2] ———, *Corrigendum: Implementation of a lattice method for numerical multiple integration*, ACM Trans. Math. Software **20** (1994), no. 2, 245.

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