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SYMPLECTIC REDUCTION AND RIEMANN-ROCH FORMULAS FOR MULTIPLICITIES

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Introduction

It is well-known that the presence of conserved quantities in a Hamiltonian dynamical system enables one to reduce the number of degrees of freedom of the system. This technique, which goes back to Lagrange and was treated in a modern spirit in papers of Marsden and Weinstein [17] and Meyer [21], is nowadays known as *symplectic reduction*.

In their paper [8] Guillemin and Sternberg considered the problem: what is the quantum analogue of symplectic reduction? In other words, when one quantizes both a mechanical system with symmetries and its reduced system, what is the relationship between the two quantum-mechanical systems that one obtains?

Recently a number of authors have made substantial progress in solving this problem, on which I shall report in this note. This development was brought about by work of Witten [27] and subsequent work of Jeffrey and Kirwan [11], Kalkman [13] and Wu [29] on cohomology rings of symplectic quotients. Another important idea turned out to be Lerman's technique of symplectic cutting or equivariant symplectic surgery [16], a generalization of the notions of blowing up and symplectic reduction.

1. Statement of the problem

The natural mathematical framework of Hamiltonian dynamics is symplectic geometry. A *symplectic form* on a smooth (C^∞) manifold M is a smooth closed two-form ω on M which is non-degenerate in the sense that at every point m the alternating bilinear form ω_m on the tangent space T_mM is non-degenerate. By elementary linear algebra, a symplectic form can exist on M only if M is even-dimensional. A *symplectic manifold* is a pair (M, ω), where M is a manifold and

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ω a symplectic form on M. The simplest example of a symplectic manifold is the phase space $\mathbf{R}^{2n} = \mathbf{R}^n \times \mathbf{R}^n$ of elementary mechanics, whose symplectic form is the constant symplectic form given in coordinates $q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n$ by

$$\omega = \sum_k dq_k \wedge dp_k. \tag{1}$$

A theorem of Darboux states that near every point on a symplectic manifold one can find a system of local coordinates $(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n)$ (called Darboux coordinates) in which the symplectic form has the simple form (1). Thus the only local symplectic invariant is the dimension, and most symplectic phenomena of interest are global in nature.

Let f be a smooth (C^∞) function on a symplectic manifold (M, ω). The *Hamiltonian vector field* (or *symplectic gradient*) of f is the unique vector field Ξ_f on M satisfying $\omega(\Xi_f, v) = df(v)$ for all vector fields v. This is well-defined because the form ω is non-degenerate. The flow of the vector field Ξ_f is called the *Hamiltonian flow* of the function f. In a system of Darboux coordinates the Hamiltonian flow of f is given by the well-known equations of Hamilton:

$$\frac{dq_k}{dt} = \frac{\partial f}{\partial p_k} \quad \text{and} \quad \frac{dp_k}{dt} = -\frac{\partial f}{\partial q_k} \qquad \text{for } k = 1, \ldots, n.$$

For example, if $M = \mathbf{R}^{2n}$ and $f(q,p) = p_k$, the Hamiltonian flow of f is a uniform translation along the q_k-axis, and if $f(q,p) = (q_k^2 + p_k^2)/2$, the Hamiltonian flow of f is a uniform rotation of period 2π in the (q_k, p_k)-plane.

Quantization refers to a rule Q assigning to every symplectic manifold M a Hilbert space $Q(M)$ and to each observable quantity (that is, a function) on the manifold a selfadjoint operator $Q(f)$ on the Hilbert space in such a manner that the following conditions are satisfied. Firstly, the operator $Q(f)$ should depend linearly on the function f; secondly, to every constant function should be assigned the corresponding multiplication operator on the Hilbert space. The third condition is Dirac's commutator condition, which says that

$$\big[Q(f), Q(g)\big] = iQ\big(\{f, g\}\big), \tag{2}$$

where $\{f, g\} = \omega(\Xi_f, \Xi_g)$ denotes the Poisson bracket of the functions f and g.

The fourth and last condition is an irreducibility condition, which can be phrased as follows. Consider a finite-dimensional subspace $\mathfrak{g}$ of the space of smooth functions on a symplectic manifold M. Let us assume that $\mathfrak{g}$ is closed under the Poisson bracket, so that it forms a Lie algebra. Let us also assume that the Hamiltonian vector fields of the functions in $\mathfrak{g}$ are complete. Then the Hamiltonian flows of the functions in $\mathfrak{g}$ generate an action on M of a connected finite-dimensional Lie group G with Lie algebra $\mathfrak{g}$. The transpose of the inclusion map $i\colon \mathfrak{g} \to C^\infty(M)$, the map $\Phi\colon M \to \mathfrak{g}^*$ defined by $\Phi(m)(\xi) = i(\xi)(m)$, is called a *momentum map* for the action of G. This notion generalizes the notions of linear momentum (where $M = \mathbf{R}^n \times \mathbf{R}^n$ and $G = \mathbf{R}^n$ is the group of translations acting on the first factor) and angular momentum (where $M = \mathbf{R}^3 \times \mathbf{R}^3$ and $G = \mathrm{SO}(3)$ is the group of rotations acting diagonally). Because the action is generated by Hamiltonian vector fields, we say that M is a *Hamiltonian G-manifold.* Now note that (2) says that $-iQ$ is a homomorphism from the Lie algebra of functions (with the Poisson bracket) to the Lie algebra of anti-selfadjoint operators on the Hilbert space $Q(M)$. The operators $-iQ(\xi)$ for $\xi \in \mathfrak{g}$ therefore define a representation of the Lie algebra $\mathfrak{g}$ on $Q(M)$. Under some mild assumptions this representation can be integrated to a unitary representation of G on $Q(M)$. The irreducibility condition requires that if G acts transitively on M, then the representation $Q(M)$ should be irreducible.

For instance, let M be the phase space $\mathbf{R}^{2n}$, on which the Poisson bracket is given by

$$\{f,g\} = \sum_k \left(\frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k} - \frac{\partial g}{\partial p_k}\frac{\partial f}{\partial q_k} \right).$$

Let $\mathfrak{g}$ be the subspace spanned by the constant function 1 and the coordinate functions $q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n$. Clearly, $\mathfrak{g}$ is closed under the Poisson bracket, and the corresponding group G (known as the Heisenberg group) contains the full group of translations $\mathbf{R}^{2n}$ and therefore acts transitively. For the functions in $\mathfrak{g}$, Dirac's condition (2) specializes to $\bigl[Q(p_k), Q(q_l)\bigr] = i\delta_{kl}$, which is the mathematical expression of Heisenberg's uncertainty principle. This relation is satisfied if we take $Q(\mathbf{R}^{2n})$ to be Schrödinger's quantization, the space of square-integrable functions in the q-coordinates. Here to the functions q_k are assigned the operators "multiplication by q_k" and to the functions p_k the differential operators $-i\partial/\partial q_k$. The representation of $\mathfrak{g}$ defined in this manner can be integrated to a unitary representation of G on $Q(\mathbf{R}^{2n})$. The Stone-von Neumann theorem asserts that this is indeed an irreducible representation and that it is, up to equivalence, the only one satisfying Heisenberg's uncertainty principle.

The trouble with the foregoing is that it has been known for a long time that a quantization scheme Q with the four properties discussed above does not exist. In the example of $\mathbf{R}^{2n}$ it was shown by Van Hove that with the operators $Q(q_k)$ and $Q(p_k)$ being defined as above a consistent assignment $f \mapsto Q(f)$ cannot be made whenever f contains terms of order greater than two in the ps and qs. To resolve this problem, over the past decades several amendments to the rules of quantization have been proposed. Most of these aim at finding a suitable subclass of the class of all functions that can be quantized in a consistent fashion and depend upon a choice of some auxiliary structure on the manifold and therefore work well for special types of symplectic manifolds only. One of the most successful is the Kirillov-Kostant-Souriau theory of geometric quantization, a good reference for which is Woodhouse's book [28]. Within the framework of these theories some rather limited uniqueness results are known, which assert that in special situations the result of the quantization process is to some extent independent of the choice of the auxiliary data. Despite these difficulties, it has turned out to be fruitful from a heuristic point of view to pretend that a quantization scheme exists that is more or less unique and *grosso modo* obeys the four rules set forth above. Typically, one expects the quantization of a non-compact manifold to be infinite-dimensional, whereas the quantization of a compact manifold ought to be finite-dimensional.

With this caveat in mind, let us now consider an arbitrary Hamiltonian G-manifold M with momentum map Φ. The technique of symplectic reduction referred to in the introduction consists of two steps: first one imposes constraints by setting each of the functions in $\mathfrak{g}$ equal to zero, and then one factors the constraint set by the action of the group G. The resulting topological space, $\Phi^{-1}(0)/G$, is called the *reduced phase space* or *symplectic quotient* of M and is denoted by M_0. We shall assume that M_0 is a smooth manifold; it then carries a natural symplectic form ω_0 induced by the symplectic form on M. (A sufficient condition for M_0 to be a manifold is that the action of G on the constraint set $\Phi^{-1}(0)$ be proper and free.)

This situation arises in classical mechanics when one considers a Hamiltonian vector field Ξ_f on M possessing a number of integrals of motion, that is, a Lie algebra $\mathfrak{g}$ of functions that Poisson-commute with f. The point of the construction is that the Hamiltonian flow of f is invariant under the group G and descends to a Hamiltonian flow on the symplectic quotient M_0.

A surprising link between this set-up and geometric invariant theory was discovered by Guillemin and Sternberg [8], Kirwan [14] and others, who found that if M is a nonsingular complex projective variety, M_0 can be identified with a geometric quotient in the sense of Mumford. We shall have nothing further to say about Hamiltonian dynamics or geometric invariant theory, but instead focus on the relationship between reduction and quantization.

After these preliminaries we can now state the reduction-versus-quantization problem. It was formulated by Guillemin and Sternberg in approximately the following terms:

"Theorem" . *Let M be a Hamiltonian G-manifold with momentum map Φ, and let $M_0 = \Phi^{-1}(0)/G$ denote the symplectic quotient of M. Then there is an isomorphism of Hilbert spaces,*

$$Q(M_0) \cong Q(M)^G, \tag{3}$$

where $Q(M)^G$ denotes the subspace of G-fixed vectors in the representation $Q(M)$.

This assertion is sometimes loosely paraphrased by saying that quantization commutes with reduction. It is possible to reduce M at values $\lambda \in \mathfrak{g}^*$ different from 0, and for these values there is an analogous statement involving the isotypical subspaces of $Q(M)$, but in this note I shall only consider the case $\lambda = 0$, which is the most important one.

In view of the non-existence of the quantization functor Q, Guillemin and Sternberg's "theorem" is to be thought of as a guiding principle, which one can only hope to make sense of by judiciously bending the rules or by imposing sufficiently strong hypotheses. (This is the reason why they called it a "theorem", not a conjecture.) Consider for example $M = \mathbf{R}^{2n}$ and the Lie algebra $\mathfrak{g}$ generated by the single function p_1. Then G is the group of translations along the q_1-axis, so that $M_0 = \mathbf{R}^{2(n-1)}$ and $Q(M_0)$ is the space of square-integrable functions in $q_2, q_3, \dots, q_n$. On the other hand, the space $Q(M)^G$, being the space of square-integrable functions in $q_1, q_2, \dots, q_n$ that are annihilated by $-i\partial/\partial q_1$, is $\{0\}$. Although the statement of the "theorem" is not strictly correct, we can simply force it to be correct by slightly enlarging $Q(M)$ before taking the G-invariants.

In the context of geometric quantization, rigorous versions of the "theorem" have been stated and proved by Guillemin and Sternberg [8] for Kähler manifolds, by Gotay [5] for cotangent bundles, and in an infinite-dimensional example of interest in gauge theory by Axelrod et al. [1]. A general procedure for quantizing a reduced system is the BRST method, which was put on a mathematical footing by Kostant and Sternberg [15] and by means of which the "theorem" was verified in a number of examples by Duval et al. [4]. (They also found a physically interesting example where it is false.) For a version of the "theorem" in the setting of microlocal analysis, see [9].

2. Quantization as an equivariant index

A definition of quantization that is apparently due to Bott has recently turned out to lead to particularly elegant versions of the "theorem" in the case where both the symplectic manifold M and the group G are *compact*. The remainder of this note will be devoted to these new results. An underlying assumption, which is common to most quantization schemes, is that the symplectic form ω is integral in the sense that for every integral cycle c on M the period $\int_c \omega$ is an integer. By the Chern-Weil theory of connections and curvature this implies that there exists a complex line bundle L on M with a Hermitian fibre metric such that the curvature form of L is equal to ω. Bott's idea was to define $Q(M)$ as a push-forward in K-theory:

$$Q(M) = \pi_*\big([L]\big),$$

where $[L]$ denotes the class of L in the K-theory of M and π is the map sending M to a space consisting of one point. In this approach $Q(M)$ is an element of the K-theory of a point, that is, a virtual vector space. This has the disadvantage that the dimension of $Q(M)$ may be negative and also that there is no natural choice of an inner product. But this is more than compensated for by the fact that the dimension of $Q(M)$ is equal to the index of a rolled-up Dolbeault complex, which by the Atiyah-Singer index formula is given by an integral of explicit differential forms on M. An additional advantage over some earlier approaches is that M is not required to possess a complex structure or any other type of polarization. Apart from the symplectic form, the only thing needed is a compatible almost complex structure, which always exists, and the dimension of $Q(M)$ is independent of it. We shall call the space $Q(M)$ the *almost complex quantization* of M.

How far does almost complex quantization comply with the rules discussed in the previous section? Consider a compact connected group G acting on M in a Hamiltonian fashion. In favourable circumstances the action of G can be lifted to an action on L by bundle transformations. (For Hamiltonian actions, there always exists a natural lift of the action of the Lie algebra $\mathfrak{g}$ to an action of $\mathfrak{g}$ on L; see e. g. [8]. Under a mild topological condition this action can then be integrated to an action of G.) In other words, L defines a class in the *equivariant* K-theory of M, and $Q(M)$ is therefore an element of the equivariant K-theory of a point, that is, a virtual representation of G. Infinitesimally, we have a virtual representation of the Lie algebra $\mathfrak{g}$; the functions in $\mathfrak{g}$ are thus quantized as operators $Q(f)$ on the virtual vector space $Q(M)$. Moreover, if the action on M is transitive, it can be deduced from the Borel-Weil-Bott theorem that $Q(M)$ is an irreducible representation.

Furthermore, by the equivariant index theorem the character of the virtual representation $Q(M)$ associated to a Hamiltonian G-manifold M can be calculated as an integral of explicit equivariant differential forms on M. This gives some hope that a detailed comparison between the equivariant index formula on the manifold M and the ordinary index formula on the symplectic quotient M_0 might lead to a proof of the "theorem".

This hope was fulfilled by the work of Vergne [23, 24, 26], Guillemin [7], Jeffrey and Kirwan [10, 12], Duistermaat et al. [3], and Meinrenken [18, 19, 20].

The source of this development was a formula proposed by Witten in [27] known as the nonabelian (or more correctly "not necessarily abelian") localization formula. The formula expresses integrals over M of G-equivariant differential forms in terms of data localized near the critical points of the norm square of the momentum map. One of its purposes is to determine cup products of cohomology classes on the symplectic quotient M_0 in terms of data on M.

In [11] Jeffrey and Kirwan proved a version of Witten's localization formula expressing integrals of equivariant forms over M in terms of integrals not over the critical points of the norm square of the momentum map, but over the components of the fixed point set of the maximal torus T of G. As a corollary they obtained a formula, the residue formula, which calculates integrals of forms over the quotient M_0 in terms of integrals of certain equivariant forms over the T-fixed points in M. In the special case of a circle action Witten's localization formula was proved by Kalkman [13] and Wu [29].

The index theorem says that the dimension of $Q(M_0)$ can be computed by evaluating the cohomology class $Ch(L_0)\, Td(M_0)$ on the fundamental class of M_0. Here $Ch(L_0)$ denotes the Chern character of L_0, the Hermitian line bundle on the quotient M_0 whose curvature form is equal to the reduced symplectic form ω_0, and $Td(M_0)$ denotes the Todd class of M_0. The Chern character $Ch(L_0)$ is represented by the differential form $\exp \omega_0$, which appears naturally in the Witten-Jeffrey-Kirwan localization formula. For this reason, a promising way to tackle the quantization problem is to apply the localization formula to the Todd class of M_0.

This is essentially the approach taken in the papers [7, 23, 26, 18] and [12], leading to several proofs of the "theorem" valid under various hypotheses, which I shall not state in detail here. In brief, in [7] Guillemin used the localization formula to reduce the proof of (3) to the verification of a combinatorial identity involving lattice points in polytopes, which he observed to be true in the case of a quasi-free torus action. In [18] Meinrenken extended these ideas to the case of a general torus action. His paper also contains a proof of an asymptotic version of (3) valid for arbitrary compact groups. In [23, 26] Vergne independently gave two proofs of (3) for torus actions, one based on localization formulas in equivariant cohomology, the other on the index theorem for transversally elliptic operators. (Note however that her definition of quantization differs from ours in that she twists the quantizing line bundle by the square root of the canonical line bundle.) In [12] Jeffrey and Kirwan used their residue formula to give a proof of (3) for groups of rank one under a mild hypothesis on the critical values of the momentum map.

A seemingly very different method was used in the papers [3] and [19] based on Lerman's technique of symplectic cutting. The two approaches are however not unrelated, since it is possible to give a proof of Jeffrey and Kirwan's residue formula using symplectic cutting; see [16]. Duistermaat et al. [3] gave a simple proof of (3) for circle actions, whereas Meinrenken's paper [19] contains the first proof valid for general compact groups.

More recently, Jeffrey and Kirwan [10] have also obtained a proof valid in this generality based on the residue formula. However, their result neither follows from nor implies Meinrenken's: on one hand they relax the condition of positivity of the quantizing line bundle imposed by Meinrenken; on the other hand they need to impose a condition stating roughly that 0 should be not too close to the singular values of Φ.

In all of the above papers the assumption is made that 0 is a regular value of the momentum map. Under this assumption the reduced space is usually not a manifold, but an orbifold, a space with finite-quotient singularities. It is possible to extend the definition of quantization given above to the category of symplectic orbifolds.

In the sequel I shall present a sketch of a proof of (3) following the ideas of [3] and [19], where I shall for simplicity assume that G is a torus.

3. The equivariant index and symplectic cutting

3.1. **Reduction and quantization.** Let M be a Hamiltonian G-manifold, where G is a torus, and let $\Phi\colon M \to \mathfrak{g}^*$ be the momentum map. We assume that M is compact and that the symplectic form on M is integral. We also assume that the symplectic quotient M_0 is a smooth manifold. Instead of the virtual representation $Q(M)$, it is more convenient to consider its character, which we shall denote by $RR(M)$. We denote by $RR(M)^G$ the integral of $RR(M)$ over G, that is, the dimension of the space of G-invariants $Q(M)^G$. Similarly, we let $RR(M_0)$ be the dimension of the virtual vector space $Q(M_0)$. Our goal is to outline a proof of the following version of Guillemin and Sternberg's "theorem":

Theorem 3.1. *$RR(M)^G = RR(M_0)$.*

The first step is to prove a special case of this statement. By a theorem of Atiyah, Guillemin and Sternberg, the image of Φ is a convex polytope, which we shall denote by Δ. Furthermore, the preimage of a vertex of Δ consists of a single component of the fixed-point set M^G. Consequently, if 0 is a vertex of Δ, the symplectic quotient M_0 is identical to the fibre $\Phi^{-1}(0)$ and is a smooth symplectic submanifold of M.

Theorem 3.2 ([20])**.** *If 0 is not contained in Δ, then $RR(M)^G = 0$. If 0 is a vertex of Δ, then $RR(M)^G = RR(M_0)$.*

See Section 3.2 for a discussion of the proof of this result.

The next step is to reduce the general case to the case where 0 is a vertex. The idea is to subdivide Δ into smaller polytopes, each of which corresponds roughly speaking to a G-invariant symplectic submanifold of M, and to prove a gluing formula for the equivariant index akin to the gluing formula for the topological Euler characteristic.

More explicitly, let $\mathcal{C}$ be a *fan* in $\mathfrak{g}^*$, that is, a finite collection of strictly convex polyhedral cones in $\mathfrak{g}^*$ satisfying the following conditions:

- $\{0\} \in \mathcal{C}$;
- if C_1 and C_2 are in $\mathcal{C}$, then $C_1 \cap C_2$ is in $\mathcal{C}$;
- if $C_1 \in \mathcal{C}$ and C_2 is a face of C_1, then $C_2 \in \mathcal{C}$.

In addition, we require that $\mathcal{C}$ be *complete* (the union of all cones in $\mathcal{C}$ is the whole of $\mathfrak{g}^*$), *rational* (every one-dimensional cone in $\mathcal{C}$ contains a nonzero integral weight), and *simplicial* (the set of one-dimensional faces of every cone in $\mathcal{C}$ is linearly independent).

For a generic choice of such a fan $\mathcal{C}$ it is possible to define a collection of compact Hamiltonian G-manifolds M_C with momentum maps Φ_C having the following properties:

1. $\Phi_C(M_C) = C \cap \Delta$, so if $0 \in \Delta$, then 0 is a vertex of the polytope $\Phi_C(M_C)$;
2. $\Phi_C^{-1}(0)$ is symplectically isomorphic to M_0;

and to prove the following formula.

Theorem 3.3 (gluing formula, [19]).

$$(-1)^{\dim\Delta} RR(M) = \sum_{C\in\mathcal{C}} (-1)^{\dim C\cap\Delta} RR(M_C).$$

See Section 3.3 for the definition of the spaces M_C and some remarks on the proof of the gluing formula.

By Theorem 3.2 and properties (1) and (2) above, $RR(M_C)^G = RR(M_0)$ for every cone C. Consequently, by Theorem 3.3,

$$(-1)^{\dim\Delta} RR(M)^G = \sum_{C\in\mathcal{C}} (-1)^{\dim C\cap\Delta} RR(M_0).$$

Moreover, $\sum_{C\in\mathcal{C}}(-1)^{\dim C\cap\Delta} = (-1)^{\dim\Delta}$, because the fan $\mathcal{C}$ is complete. This proves Theorem 3.1.

3.2. **The index formula.** By the equivariant index theorem of Atiyah, Segal and Singer the character $RR(M)\colon G\to\mathbf{C}$ can be written as a sum

$$RR(M) = \sum_{F\in\mathcal{F}} \chi_F, \tag{4}$$

where $\mathcal{F}$ denotes the collection of components of the fixed point set M^G, and where each of the contributions χ_F is a rational function on G given by an integral of equivariant differential forms on F as follows. Choose a G-invariant almost complex structure on M compatible with the symplectic form. Then the tangent bundles of M and of F, and hence the normal bundle N_F of F in M, are complex vector bundles. If $\xi\in\mathfrak{g}$ is generic in the sense that it generates a dense one-parameter subgroup of G, then

$$\chi_F(\exp\xi) = e^{i\Phi^\xi(F)} \int_F \frac{e^\omega\, Td(F)}{D_F(\xi)}. \tag{5}$$

Here $\Phi^\xi(F)$ is the constant value of the function Φ^ξ on the submanifold F, $Td(F)$ is a form representing the Todd class of F, and $D_F(\xi)$ is the equivariant form

$$D_F(\xi) = \prod_j \bigl(1-\exp(2\pi i\alpha_{jF}(\xi)+c_{jF})\bigr). \tag{6}$$

The meaning of c_{jF} and α_{jF} is as follows. One formally splits the normal bundle of F in M into a sum of complex G-equivariant line bundles: $N_F = E_{1F}\oplus E_{2F}\oplus\cdots\oplus E_{rF}$, where $r=\operatorname{rank} N_F$, and defines $c_{jF}\in\Omega^2(F)$ to be the Chern form of E_{jF} and $\alpha_{jF}\in\mathfrak{g}^*$ the weight of the G-action on the fibre of E_{jF} at any point of F. (The c_{jF} and the α_{jF} may not be well-defined, but any symmetric function of them is, such as the form $D_F(\xi)$.)

On the other hand we can write the character as a finite sum

$$RR(M)(\exp\xi) = \sum_{\lambda\in\Lambda} N(\lambda) e^{2\pi i\lambda(\xi)}, \tag{7}$$

where Λ is the lattice of integral weights in $\mathfrak{g}^*$ and the $N(\lambda)$ are integers.

Now assume that 0 is not contained in the polytope Δ, or that it is a vertex of Δ. Let C be the smallest convex cone with vertex 0 containing Δ. Choose the vector ξ such that $\mu(\xi)\geq 0$ for all $\mu\in C$. In (4) and (7) we can substitute $it\xi$ for ξ, where t is real, and study the asymptotics as $t\to\infty$.

If the fixed-point component F is different from $\Phi^{-1}(0)=M_0$, then $\Phi^\xi(F)>0$, so (5) implies that $\chi_F(\exp it\xi) = O(t^{-\infty})$ for $t\to\infty$.

If 0 is not in Δ, we infer from this that $RR(M)(\exp it\xi) = O(t^{-\infty})$ as $t \to \infty$. This means that the constant term $N(0) = RR(M)^G$ in the Fourier expansion (7) vanishes. This proves the first assertion of Theorem (3.2).

If 0 is a vertex of Δ, then 0 is the minimum value of Φ^ξ, and therefore $\alpha_{jF}(\xi) > 0$ for $F = M_0$, and hence $\chi_{M_0} = \int_{M_0} e^\omega \, Td(M_0) + O(t^{-\infty})$. Applying the Riemann-Roch theorem to M_0 and adding all the terms χ_F, we obtain

$$RR(M)(\exp it\xi) = RR(M_0) + O(t^{-\infty}) \tag{8}$$

as $t \to \infty$. In particular, $RR(M)(\exp it\xi)$ is bounded for large t. Comparing this with (7), we see that $N(\lambda) = 0$ if λ is not in C (because for generic ξ the exponents $2\pi i\lambda(\xi)$ are all distinct), and therefore

$$RR(M)(\exp it\xi) = N(0) + O(t^{-\infty}) \tag{9}$$

as $t \to \infty$. The estimates (8) and (9) immediately imply the second part of Theorem 3.2.

3.3. **The gluing formula.** The notation and conventions are as in Section 3.1. For every cone C in the fan $\mathcal{C}$ let $\langle C \rangle$ denote the linear span of C. Let $\mathfrak{g}_C$ be the annihilator in $\mathfrak{g}$ of $\langle C \rangle$, and let G_C be the analytic subgroup $\exp \mathfrak{g}_C$ of G. The *symplectic cut* of M with respect to C is defined by induction on the dimension of C:

$$M_C = \Phi^{-1}(\mathring{C})/G_C \cup \bigcup_{C' < C} M_{C'},$$

where $\mathring{C}$ denotes the relative interior of C and the union is over all faces C' of C. The map $\Phi_C \colon M_C \to \mathfrak{g}^*$ is obtained by restricting Φ to $\Phi^{-1}(C)$ and pushing it down to M_C. The image of Φ_C is equal to $C \cap \Delta$. Also, G acts in a natural way on M_C with the subgroup G_C acting trivially.

Clearly, $M_{C'}$ is a subset of M_C if $C' < C$. The subset $\Phi^{-1}(\mathring{C})/G_C$ is open and dense in M_C. If $C = \{0\}$, M_C is simply the symplectic quotient M_0. If C is top-dimensional ($\dim C = \dim G$), then the open dense piece is $\Phi^{-1}(\mathring{C})$ and is equivariantly diffeomorphic to an open piece of M.

It can be shown that if the fan $\mathcal{C}$ is generic with respect to the polytope Δ, then for every cone C the space M_C is a symplectic manifold and the G-action is Hamiltonian with momentum map Φ_C. Here "generic" means, among other things, that for every fixed-point component F of M the image $\Phi(F)$ has to be in a top-dimensional cone of $\mathcal{C}$.

(Here I have slurred over an interesting but somewhat technical complication: even for generic fans the M_C are usually not manifolds, but orbifolds. For the sake of simplicity I shall ignore this problem and pretend they are manifolds. This gap in our proof of Theorems 3.1–3.3 can be filled by invoking index theorems for orbifolds due to Atiyah, Kawasaki and Vergne. See e. g. [25].)

Finally, a word on the proof of the gluing formula, Theorem 3.3. Let us denote the set of fixed-point components in M_C by $\mathcal{F}_C$ and write the equivariant index of M_C as a sum of fixed-point contributions as in (4): $RR(M_C) = \sum_{F \in \mathcal{F}_C} \chi_{C,F}$. The gluing formula then amounts to the identity

$$(-1)^{\dim \Delta} \sum_{F \in \mathcal{F}} \chi_F = \sum_{C \in \mathcal{C}} (-1)^{\dim C \cap \Delta} \sum_{F \in \mathcal{F}_C} \chi_{C,F}. \tag{10}$$

To evaluate the right-hand side, we note that the fixed-point components of the cut manifolds M_C fall into two classes:

1. $F \subset M_C$ where $\dim C = \dim G$ and $\Phi_C(F) \in \mathring{C}$. These are the fixed-point components coming from fixed-point components in M. Indeed, since $\Phi_C^{-1}(\mathring{C})$ is isomorphic to $\Phi^{-1}(\mathring{C})$, F is an isomorphic copy of a fixed-point component in M, which we shall also denote by F, and $\chi_{C,F} = \chi_F$. Consequently the sum $\sum_F \chi_{C,F}$ over all these Fs is equal to $\sum_{F \in \mathcal{F}} \chi_F = RR(M)$.

2. $F \subset M_C$ for some C with $\dim C < \dim G$. These are fixed-point components that do not correspond to fixed-point components in M, but are introduced by the cutting process. An isomorphic copy of F, which we shall also denote by F, occurs in every $M_{C'}$ such that $\Phi(F) \in C'$. To finish the proof of (10) it suffices to show that for all such F

$$\sum_{\substack{C \\ \Phi(F) \in C}} (-1)^{\dim C \cap \Delta} \chi_{C,F} = 0. \tag{11}$$

Let me briefly indicate the proof of this identity in the important special case $F = M_0$, the fixed-point component that is contained in M_C for every $C \in \mathcal{C}$. Using (5) we see that in this case (11) amounts to

$$\sum_{C \in \mathcal{C}} (-1)^{\dim C \cap \Delta} \frac{1}{D_{C,M_0}(\xi)} = 0, \tag{12}$$

where D_{C,M_0} is the equivariant form defined as in (6) associated to the normal bundle N_C to M_0 in M_C. In fact,

$$D_{C,M_0}(\xi) = \prod_j \bigl(1 - \exp\bigl(2\pi i \lambda_j(\xi) + \lambda_j(c)\bigr)\bigr), \tag{13}$$

where $\lambda_j \in \Lambda$ are the generators of the one-dimensional faces of C, and $c \in \Omega^2(M_0, \mathfrak{g})$ is the curvature form of the principal G-bundle $\Phi^{-1}(0) \to M_0$.

The identity (12) boils down to a purely combinatorial identity involving the fan $\mathcal{C}$. Take an element ξ of the complexified Lie algebra $\mathfrak{g}^{\mathbf{C}}$ and an integral weight λ. Then $z = \exp \xi$ is an element of the complexified torus $G^{\mathbf{C}}$ and it makes sense to define $z^\lambda = \exp 2\pi i \lambda(\xi)$. To each cone C we associate the rational function f_C on $G^{\mathbf{C}}$ defined by $f_C(z) = \prod_j (1 - z^{\lambda_j})^{-1}$. A multi-variable generalization of the identity $(1-z)^{-1} + (1-z^{-1})^{-1} = 1$ says that

$$\sum_{C \in \mathcal{C}} (-1)^{\dim C} f_C = 0.$$

Substituting the equivariant form $\exp\bigl(2\pi i \lambda_j(\xi) + \lambda_j(c)\bigr)$ for z^{λ_j} and using (13), we obtain (12).

3.4. **Further developments.** If 0 is not a regular value of the momentum map Φ, the quotient M_0 is usually not a manifold or even an orbifold, but contains serious singularities. It is of interest to extend Theorem 3.1 to this situation. Recent joint work of Meinrenken and the author based on the methods outlined in this note shows that this is indeed possible. (Previously, a version of Theorem 3.1 for singular quotients was obtained in [22] under the hypothesis that M admits a compatible complex structure.)

The statement of Theorem 3.1 can be made sense of in the setting of presymplectic and Spin$^{\mathbf{C}}$-manifolds and was proved for completely integrable torus actions on such manifolds by Grossberg and Karshon [6]. Further results in this direction have been announced by Canas da Silva et al. [2].

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