BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 3, July 1996

Commutative algebra with a view toward algebraic geometry, by David Eisenbud, Graduate Texts in Math., vol. 150, Springer-Verlag, Berlin and New York, 1995, xvi+785 pp., $69.50, ISBN 0-387-94268-8

In the title of a story [To] written in 1886, Tolstoy asks, "How Much Land Does a Man Need?" and answers the question: just enough to be buried in. One may ask equally well, "How much algebra does an algebraic geometer—man or woman—need?" and prospective algebraic geometers have been known to worry that Tolstoy's answer remains accurate. Authors of textbooks have at times given vastly different answers, albeit with a general tendency over time to be monotonely increasing in what they expect will be useful. In crudely quantitative terms, one has at one extreme the elegant minimalist introduction of Atiyah and Macdonald [AM] at 128 pages, then Nagata [Na] at 234 pages, Matsumura [Ma] at 316 pages, Zariski and Samuel [ZS] at 743 pages, and the present work at 785 pages.

I suspect that most of my fellow algebraic geometers, as a practical matter, would answer my question by saying, "Enough to read Hartshorne." Hartshorne's *Algebraic geometry* [Ha], appearing in 1977, rapidly became the central text from which recent generations of algebraic geometers have learned the essential tools of their subject in the aftermath of the "French revolution" inspired by the work of Grothendieck and Serre (*Principles of algebraic geometry* by Griffiths and Harris [GH] plays a comparable role on the geometric side for the infusion of complex analytic techniques entering algebraic geometry at about the same period). Hartshorne's book is peppered with references to the then-existing texts in commutative algebra, no one of which contained everything he needed. Indeed, one of the goals of Eisenbud's book was to provide a single work containing all of the algebraic results needed in Hartshorne's book.

It is sobering to consult Hodge and Pedoe's *Methods of algebraic geometry* [HP] and see how little commutative algebra they got away with—for example, there is no entry for "Noetherian" in the index. The student reader (I was one) was set loose in algebraic geometry armed with not much more than the fundamental theorem of algebra, resultants, the Hilbert basis theorem, the Nullstellensatz, and the Plücker equations for the Grassmannian (which sneaks in a bit of representation theory), plus some splendid geometric insights. It is heartening to feel that algebraic geometry has attained the point where we have at our disposal the power of scheme-theoretic techniques without losing the inexhaustible wellspring of inspiration supplied by geometric insights (these insights were kept alive most vividly in the work of Griffiths and Mumford).

It is interesting to observe that about the time that Hartshorne's book was being published, a second wave of algebraic input into algebraic geometry, admittedly less powerful than the first, was gathering strength. Rather than being insights having universal application throughout algebraic geometry, such as Serre's FAC [Se], they tend to be more specific and to involve more specialized algebra. The discovery by Kempf [Ke] and Kleiman-Laksov [KL] that the singularities of the theta-divisor of the Jacobian variety of an algebraic curve may be studied using

1991 *Mathematics Subject Classification.* Primary 13-01, 14-01, 13A50, 13C15.

determinantal varieties (varieties defined by the minors of matrices whose entries are polynomials) might serve as an opening of this phase of the influence of algebra on algebraic geometry. About the same time, a theorem of Macaulay about what we would now call Gorenstein rings appears as the crucial step in Griffiths' proof [Gr] that the derivative of the period map for projective hypersurfaces is injective (infinitesimal Torelli). The beautiful use of the Eagon-Northcott resolution in the bound by Gruson-Lazarsfeld-Peskine [GLP] for the regularity of ideals of projective curves is another example. Eisenbud's book has the great virtue of incorporating the commutative algebra that lies behind this second wave of algebraic influence on algebraic geometry as well.

There are certainly topics in algebra that lie outside the scope of Eisenbud's book which belong in the arsenal of many algebraic geometers. Representation theory in many guises appears in algebraic geometry, for example, in the geometry of the period domains which appear in Hodge theory, in geometric invariant theory, or the various more specialized uses of the representation theory of the general linear group, such as Kempf's derivation of the Eagon-Northcott complex. A student planning to work in arithmetic algebraic geometry might wish for a more abstractly oriented package of tools—simplicial objects, the derived category, algebraic K-theory. It should be noted that Tolstoy's protagonist collapses in the attempt to include just one additional tract of ground, and Eisenbud has managed to distill the gist of some of these topics into a series of highly concentrated appendices.

The subject of commutative algebra itself has undergone considerable changes in the period since Hartshorne's book was written—although probably not, with one exception, a revolution. There is one change which has overtaken commutative algebra that is in my view revolutionary in character—the advent of symbolic computation. This is as yet an unfinished revolution. At present, many researchers routinely use Macaulay, Maple, Mathematica, and CoCoA to perform computer experiments, and as more people become adept at doing this, the list of theorems that have grown out of such experiments will enlarge. The next phase of this development, in which the questions that are considered interesting are influenced by computation and where these questions make contact with the real world, is just beginning to unfold. I suspect that ultimately there will be a sizable applied wing to commutative algebra, which now exists in embryonic form. Eisenbud has been very much involved in computational developments; he has, for example, authored many of the basic scripts in use with Macaulay. He has included a highly useful chapter on Gröbner bases containing most of the basic theorems and with a series of suggested computational projects. I am in agreement with him that this is an area that most young algebraic geometers ought to learn.

Eisenbud's book is clearly intended to serve both as an introduction for students and as a reference work. It is difficult to harmonize these two goals, and indeed many reference works, stating theorems in maximal generality, are virtually unreadable. Eisenbud's strategy for surmounting this difficulty is quite interesting and successful. A typical chapter begins with an informal discussion, in which he attempts to explain to the reader what is really going on and why the topic is important and interesting. These discussions are almost invariably illuminated by Eisenbud's remarkable gift for producing the telling example. He then wipes the slate clean and begins again, giving formal definitions and proofs. He frequently takes the unusual step of explaining what is not true and why the theory is not simpler than it is. This is then followed by a profusion of exercises, drawn from the heartland of commutative algebra and from algebraic geometry.

The book is informed by Eisenbud's broad knowledge of algebraic geometry and his encyclopaedic knowledge of commutative algebra; he works actively in both fields. The algebra is illuminated whenever possible by its geometric interpretation, a feature that I found extraordinarily useful and one which I imagine some readers from the commutative algebra side will find enlightening. He brings the reader as close as he can to late-breaking developments in commutative algebra, with numerous references to many of the important things currently going on.

My friend Harsh Pittie once gave the following description of the styles of mathematical exposition of three of the leading mathematicians of the day: A paper or lecture by X was like a walk in a beautiful garden. Y would take you up in an airplane and show you the reservoir from which the garden ultimately got its water, while with Z you got into a jeep and went careening through the shrubbery. The expository style of this book is mostly of the "walk-in-the-garden" variety, although Eisenbud does take the reader up in a plane when necessary, and there are rare but occasional crashes through the shrubbery. The style is delightfully old-fashioned, with digressions, interesting stories, apostrophes to the reader (including one exhorting him or her to generalize a conjecture of mine), puns, historical excursions, and advice. The book is infused with an evident affection for both subjects, commutative algebra and algebraic geometry, and Eisenbud displays equal relish in showing the reader the Hilbert-Burch Theorem and the geometry of a trigonal canonical curve.

The existence of this book raises some interesting questions about how students in algebraic geometry ought to be trained. Traditional first-year graduate courses in algebra often have a rather perfunctory treatment of rings, ideals, and modules, emphasizing instead group theory and field theory. For a future algebraic geometer, field theory is essential, an introduction to representation theory would be more useful than the Sylow theorems, and a solid introduction to commutative algebra is vital. The increasing sophistication and diversity of the algebraic tools now in use in algebraic geometry require students to be quite selective if they are going to get a Ph.D. in a reasonable amount of time and make it really important to acquire the habit of lifelong learning and persistently expanding one's repertoire of mathematical techniques. My advice to a student would be to read the portions of Eisenbud's book relevant to Hartshorne, skimming where appropriate and perhaps shifting back and forth between the two books, and then to nibble further at Eisenbud's book over the succeeding years.

This volume is a major and highly welcome addition to the mathematical literature, providing a unified, elegant, and exhaustive survey of those topics in commutative algebra likeliest to be of use to algebraic geometers. The rigorous treatment is supplemented by substantial heuristic, historical, and motivational sections and a wide range of exercises. There is an exceptionally thorough bibliography and numerous links to recent developments. I anticipate that it will soon be found on the bookshelf of virtually any practicing algebraic geometer or commutative algebraist.

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