BOOK REVIEWS

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Riemannian geometry: *A modern introduction*, by Isaac Chavel, Cambridge Tracts in Mathematics, no. 108, Cambridge Univ. Press, 1994, xii + 386 pp., \$59.95, ISBN 0-521-43201-4; also in paperback as *Riemannian geometry*: *A modern introduction*, Cambridge Tracts in Mathematics, no. 108, Cambridge Univ. Press, \$24.95, ISBN 0-521-48578-9

Much of mathematics is so technical that it is appreciated by only specialists in that area. How do you convince someone, even another mathematician, that a mathematical problem or result is interesting? Traditionally, one of the simplest ways is to relate it to some aspect of geometry, number theory, or an application to physics, engineering, or the social sciences. The recent solution of the Fermat problem is an example where the use of formidable technical material has motivated many to learn that material. As another example, the technical aspects of partial differential equations are often appreciated because of the insight and tools they have given both to applied problems and, more recently, to geometry.

Geometry plays a special role as one of the more immediately appreciable parts of mathematics. It is a meeting ground for a diversity of techniques. In many ways it is applied mathematics in the deepest sense, because in attacking a geometry problem one frequently must invoke and apply techniques and intuition from all branches of mathematics.

At the birth of the central innovation of Greek mathematics, geometry was a principal protagonist. Descartes's introduction of coordinates was the marriage, sometimes resented by various purists, of algebra and geometry. Coordinates not only paved the way for a systematic calculus, it also permitted the systematic study of curves and surfaces — rather than the limited world of triangles, parallelograms, and conic sections along with their three-dimensional analogs. To understand the curvature at a point on a smooth plane curve, by a rigid motion we may assume this point is at the origin and that the $x-$axis is the tangent line. Then the Taylor expansion at the origin is

$$y = f(x) = \tfrac{1}{2}cx^2 + \cdots .$$

The constant c is the curvature of this curve ($\frac{1}{c}$ is also the radius of curvature of the osculating circle). Similarly, for a point on a smooth surface in $\mathbf{R}^3$, we may

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assume the point is at the origin and the xy plane is its tangent plane. Then the Taylor expansion has the form

$$z = f(x,y) = \tfrac{1}{2}(ax^2 + 2bxy + cy^2) + \cdots ,$$

where $a = f_{xx}(0,0)$, etc. The eigenvalues of the second derivative quadratic form $ax^2 + 2bxy + cy^2$ are the principal curvatures k_1, k_2. From these it is natural to consider the invariants: trace $H = k_1 + k_2$ and determinant $K = k_1 k_2$. These are the mean and Gauss curvatures, respectively. One can also compute the element of arc length

$$ds^2 = E(x,y)dx^2 + 2F(x,y)dx\,dy + G(x,y)dy^2,$$

and use this to compute the element of area.

Geometric problems were among the earliest considered in the calculus of variations. Arc length was used to study paths of shortest length (geodesics) on a surface, to study, among all surfaces with a given boundary, the surface having least area (minimal surfaces), and to study, among all regions in the plane whose boundary has a given length, the region with greatest area (isoperimetric problem). See Hildebrandt-Tromba [HT] for a beautiful non-technical introduction and Osserman's book [Os2] for a readable more detailed treatment.

The Euler characteristic of a surface as well as aspects of projective geometry made it clear that there are some geometric properties that do not depend on the metric properties of a surface. Gauss's classical discovery — by a long explicit computation — that the Gauss curvature K is determined by the element of arc length alone and is independent of the way the surface lies in $\mathbf{R}^3$ was the start of the study of intrinsic differential geometry. The Gauss-Bonnet Theorem was also a significant generalization of the Euclidean result that the sum of the angles of a triangle is always π, as well as relating, for compact surfaces, the curvature and the Euler characteristic. It was one of the earliest global — rather than local — theorems in differential geometry. See Osserman's wonderful book [Os1] for a non-technical introduction to the essential ideas in differential geometry, remarkably without formulas.

The next major step in the study of intrinsic geometry was Riemann's generalization of this to higher dimensions. He began with an element of arc length

$$ds^2 = \sum_1^n g_{ij}(x)dx^i\,dx^j$$

and asked how much one can simplify this by making a change of coordinates $x_i = \varphi_i(y_1, \dots , y_n)$. For instance, how could one recognize if, in some new coordinates $y = \varphi(x)$, this is just the Euclidean metric $ds^2 = \sum (dy^k)^2$? Riemann observed that at one point — which may be assumed to be at the origin — one could always pick coordinates so that $g_{ij}(0) = \delta_{ij}$ and its first partial derivatives there are zero. Thus, by a Taylor expansion

$$ds^2 = \sum_{i,j=1}^n \Big(\delta_{ij} + \sum_{k,\ell=1}^n \frac{\partial g_{ij}}{\partial x^k \partial x^\ell}(0)x^k x^\ell + \cdots\Big)dx^i\,dx^j.$$

The essential term is then $\partial g_{ij}/\partial x^k \partial x^\ell$. After brilliant scrutiny to understand its invariance properties, Riemann was led to introduce the curvature tensor $R_{ijk\ell}$ as an appropriate combination of the second derivatives $\partial g_{ij}/\partial x^k \partial x^\ell$. He showed, for instance, that one has the Euclidean metric if and only if $R_{ijk\ell} = 0$. (See Spivak [Sp, Vol. 2] for a lucid discussion.)

Christoffel observed that one should split Riemann's computation of the curvature tensor into two interesting steps: compute the "Christoffel symbols" Γ_{ij}^k that involve only the first derivatives of the metric, and then compute the curvature tensor by taking the first derivatives of the Christoffel symbols. Consequently, it was natural to investigate curvature tensors that arose from any Christoffel symbols, not just those one obtains from elements of arc length. As Levi-Civita observed, these Christoffel symbols are then viewed as defining an *affine connection*, which can be interpreted as describing how a tangent vector can be moved parallel to itself; it does not require an element of arc length.

Einstein's insight that his general theory of relativity should be written in the language of Riemannian geometry gave an enormous impetus to the study of differential geometry, since it showed that Riemann's abstract generalization actually arose in the "real" world. Here the arc length is not positive definite but is based on the indefinite quadratic form $(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - dt^2$ of the light cone. Of more recent significance (known by the late 1920's) was the awareness that Maxwell's equations can be interpreted by viewing the electromagnetic potential as defining an affine connection. Maxwell's equations then express the electromagnetic field as the curvature of this connection. This was later generalized to Yang-Mills fields and is related to the even more recent work by Sieberg-Witten.

In this century geometry has vigorously pursued many paths: algebraic and differential topology, algebraic geometry, Riemannian geometry, the geometry determined by an affine connection, the geometry of a complex manifold, and symplectic geometry. The badge GEOMETER is worn proudly by many, their work using tools drawn from all of mathematics; indeed, part of the richness of contemporary mathematics is the intertwining of mathematics (and physics) so, for instance, the spectrum of the Laplacian, which was originally studied to understand the wave equation, is now a significant ingredient in diverse situations including number theory and the lengths of closed geodesics.

Chavel's *Riemannian geometry: A modern introduction* is one of the few texts that treat some developments that have occurred since 1960. The heart of this book is in Chapters 3–7, whose main theme is *comparison inequalities*.

A typical comparison inequality is Bishop's Inequality (Chapter 3): Let $V(r)$ be the volume of a ball of radius r in a complete n-dimensional Riemannian manifold whose Ricci curvature is at least $(n-1)\kappa$. Then $V(r) \geq V_\kappa(r)$, where V_κ is the volume of a ball in a space with constant sectional curvature κ. Moreover, if equality holds for some ball, then this ball is isometric to the ball of radius r in the space of constant sectional curvature κ.

Comparison inequalities have the following ingredients, all of which are easily visible in the above example:

- An "interesting" functional, such as arc length, volume, an eigenvalue of the Laplacian, isoperimetric and Sobolev constants. Discovering a new "interesting" functional is a real achievement.
- A model space, frequently one that is simply connected and has constant curvature of some sort. The choice of a model space is critical.
- An inequality comparing the functional in the given and model space. Does equality occur only for the model space itself?

These inequalities, besides their own intrinsic interest, serve as basic tools in subsequent investigations.

Chapters 1 and 2. cover the basic concepts of Riemannian geometry. While formally these chapters are more or less self-contained, they are better viewed as a summary of material learned elsewhere.

Chapter 3: Riemannian Volume. One key result is the volume comparison theorem mentioned above. A useful long appendix to this chapter discusses the eigenvalues of the Laplacian along with some related comparison theorems (see also the author's earlier volume [Ch] and Buser's more recent book [Bu]).

Chapter 4: Riemannian Coverings. The main theme here is, for a compact manifold M, to relate the global growth of the volume $V(r)$ of a ball of radius r on the universal cover with the size of the fundamental group of M. One defines $n(\lambda)$ to be essentially the number of different words one can make in the fundamental group having length λ. Milnor's Theorem asserts that if M has nonnegative Ricci curvature, then its fundamental group grows at most polynomially, while if M has strictly negative curvature, then its fundamental group has exponential growth in the sense that $n(\lambda)$ grows exponentially. The proofs of these and related results use the comparison theorems of Chapter 3.

This chapter closes with the classical Gauss-Bonnet Theorem for two-dimensional surfaces. This seems to be independent of the main material in the same chapter; I am uncertain why the author placed it here rather than in Chapter 2.

Chapter 5: Kinematic Density. Here one uses integral geometry to discuss geodesic flow. These ideas originated in classical mechanics. A basic result is Liouville's Theorem; on the unit tangent bundle Liouville constructed a natural measure that is invariant under geodesic flow. One application is to the Berger-Kazdan comparison theorem for the volume of a compact n-dimensional manifold M with injectivity radius $\mathrm{inj}(M)$. It asserts: $\mathrm{Vol}(M) \geq c_n(\mathrm{inj}(M))/\pi$, with equality if and only if M is isometric to the standard round sphere S^n with radius $\mathrm{inj}(M)$. Here $c_n(r)$ is the volume of the standard n-sphere of radius r. If one combines this inequality with topological results of A. Weinstein (n even) and C. T. Yang (n odd), one obtains a proof of the Blaschke conjecture that the only *wiedersehen manifolds* are the standard round spheres (a complete proof for the two-dimensional case had been done earlier by L. Green).

Chapter 6: Isoperimetric Inequalities. This chapter discusses generalizations of the classical isoperimetric inequality $L^2 \geq 4\pi A$ for a region in the plane with area A and perimeter length L. The natural generalization to a region Ω in $\mathbf{R}^n$ is the comparison inequality

$$\frac{\mathrm{Area}(\partial\Omega)}{\mathrm{Vol}(\Omega)^{1-1/n}} \geq \frac{\mathrm{Area}(\mathbf{S}^{n-1})}{\mathrm{Vol}(\mathbf{B}^n)^{1-1/n}} = n\omega_n^{1/n},$$

where ω_n is the area of the standard unit ball $\mathbf{B}^n$. Note that the left side is invariant if we magnify Ω by $x \mapsto \lambda x$.

It is not at all obvious how to generalize this from the model space $\mathbf{R}^n$ to the model spaces of the round sphere or hyperbolic space. One difficulty is that these have no self-similarities generalizing the magnifications of $\mathbf{R}^n$. The author gives two approaches, one due to Gromov, the other based on the classical technique of symmetrization as extended by Figiel-Lindenstrauss-Milman.

For cases more general than the round sphere and hyperbolic space, the author presents the results of Buser, Croke, and Kanai. The subject is far from closed.

Chapter 7: Comparison and Finiteness Theorems. The basic topics here are the now classical Rauch comparison theorem and Toponogov triangle comparison theorem. This material probably would have benefited by appearing much earlier in the book. Since the author does not present any detailed applications of the Toponogov theorem, such as to the work of Grove-Shiohama, many readers may not appreciate its significance and power.

This chapter ends with Cheeger's Finiteness Theorem: consider the set of compact n-dimensional Riemannian manifolds M with $\text{diameter}(M) \leq d, \text{Volume}(M) \geq V$, and $|\mathcal{K}| \leq \kappa$; here $\mathcal{K}$ is the sectional curvature. Then there is a bound on the number of diffeomorphism classes of this set in terms of the constants n, d, V, and κ. In fact, the proof here is by contradiction, so no explicit bound is actually found. There is a more recent theorem mentioned in the notes. It was found by Grove-Peterson. They replace Cheeger's two-sided curvature assumption by only a lower bound on the sectional curvature, $\mathcal{K} \geq \kappa$, and then conclude there are finitely many homotopy types, the bound depending on the scale-invariant constants $n, d^n/V$, and κd^2.

The strength of Chavel's book is in presenting interesting current results in such a way that they are accessible to students. This book would be a splendid supplement to a text such as the books by do Carmo [doC] or Gallot-Hulin-Lafontaine [GHL] in a standard course on Riemannian geometry. It is less successful as a stand-alone introductory text, since the basic material is a bit skimpy. There are too many "one easily sees", and a substantial amount of basic material is relegated to the exercises (an instance is the whole theory of Riemannian submersions at the end of Chapter 1).

One very useful feature is the "Notes and Exercises". These are collected at the end of each chapter and give a broader, more relaxed picture of the subject. From a technical view, the book has a nice bibliography, but would have benefited from an index of notation (that would have saved me lots of time) and a more comprehensive index (Ricci curvature, co-area, isotropy group, pinching, to name a few items that I sought unsuccessfully in the index, although they are in the book).

I believe the author is setting-up a WWW page concerning this book. For more information his e-mail address is

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Note that there is a paperback version available from the publisher for about \$20 (\$24.95 less their 20% standard discount to members of the AMS), thus making it affordable to students. Readers of this book may also find it fruitful to read Peter Li's nice lecture notes [Li]. A number of the same topics are treated but somewhat differently.

References

[Bu] Buser, Peter, *Geometry and spectra of compact Riemann surfaces*, Progress in Mathematics, Birkhauser, Basel and Boston, 1992. MR **93g:**58149

[Ch] Chavel, Isaac, *Eigenvalues in Riemannian geometry*, Academic Press, New York, 1984.

[doC] do Carmo, Manfredo Perdigao, *Riemannian geometry*, Birkhauser, Basel and Boston, 1992. MR **92i:**53001

[GHL] Gallot, Sylvestre, Dominique Hulin, and Jacques Lafontaine, *Riemannian geometry*, Second Edition, Universitext, Springer-Verlag, Berlin and New York, 1990. MR **91j:**53001

[HT] Hildebrandt, S. and A. Tromba, *Mathematics and optimal form*, Scientific American Library, New York, 1985. MR **87k:**00009

[Li] Li, Peter, *Lectures on differential geometry*, Seoul University, Korea, 1991. For copies send e-mail to `pli@math.uci.edu`.

[Os1] Osserman, R., *Poetry of the universe*, Anchor, New York, 1995.

[Os2] ——, *Survey of minimal surfaces*, Van Nostrand, 1969, Second Edition published by Dover, New York, 1986. MR **41:**934; MR **87j:**53012

[Sp] Spivak, Michael, *A comprehensive introduction to differential geometry*, Vols. 1–5, Publish or Perish, Inc., Houston, TX, 2nd edition, 1979. MR **82g:**53003a; MR **82g:**53003b; MR **82g:**53003c; MR **82g:**53003d; MR **82g:**53003e

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