BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 3, July 1996

Elements of applied bifurcation theory, by Yuri A. Kuznetsov, Applied Mathematical Sciences, vol. 112, Springer-Verlag, Berlin and New York, 1995, xv+515 pp., \$59.95, ISBN 0-387-94418-4

The current spectacular interest in *dynamical systems* has gradually built up over the last forty years. Poincaré [13] set the change for the subject at the turn of the century by moving away from the analytical approach to differential equations with an emphasis on the topology of the orbit structure of the phase plane. His initial impetus was reinforced in the first half of this century by Birkhoff [7], Andronov and Pontryagin [2], and others. However, it can be argued that the real explosion of interest for dynamical systems in the mathematics community was triggered by the great contributions of Smale and Arnold in the 1960s.

The global theory of dynamical systems was developed by Smale [15] throughout the decade. His ability to focus on the essential ingredients of hyperbolic dynamics was unparalleled. That his insights would feed the frenzy of activity on chaotic behaviour and strange attractors in the following decades would have been difficult to predict. By contrast, Arnold, alongside Moser, was developing the tools to understand the special qualitative features of Hamiltonian systems and area-preserving maps. The work involved fine diophantine estimates for the persistence of invariant tori or circles under perturbation of integrable systems.

The global approach of the sixties and the attempt by the Smale school to classify a generic set of systems, which in some sense could be described simply, led to the recognition of many special types of dynamics which had degenerate structure. This in turn gave a new impetus to the work of Andronov et al. [1], Neimark, Sacker, and Hopf on families of dynamical systems. The occurrence of change of structure, or bifurcation, in families was a major route for applied workers to predict global phenomena by making local calculations. For example, the Neimark-Sacker-Hopf bifurcations predict the growth of a limit cycle or invariant circle around an equilibrium point which undergoes a change of stability. There are nondegeneracy conditions associated with the creation of this circle, but they are all calculated at the fixed point.

The applied modeller may have no idea which of the parameters are critical to unfolding the degenerate behaviour of the defining equations. Of course, some parameters may have no structural effect on the system whatsoever. Often a dynamical system modelling a real-life phenomenon comes prepackaged with too many parameters for the applied mathematician to handle—it is very daunting to search in large-dimensional parameter spaces! The growth and classification of bifurcation has been extremely valuable in allowing modellers to focus their search for key dynamical behaviour in the parameter space.

Moreover, suppose the bifurcational behaviour of a system has been understood, and a complete set of topological types of phase portrait that occur as the degenerate system unfolds by varying parameters has been found. The collection of pictures for a bifurcation can then be an immensely useful guide to the behaviour

1991 *Mathematics Subject Classification.* Primary 34A47, 35B32, 58F14, 58F39.

that can be expected in the model. In particular, it enables the search of parameter space to be carried out with greater confidence.

There have been many books produced on dynamical systems to reflect the recent interest, but relatively few of the books are sufficiently encyclopaedic to offer a wide and authoritative account of the area. In particular, there has been a sequence of texts which have gradually developed and consolidated the subject of bifurcation theory. Some recent milestones are the books by Andronov et al. [1], Marsden and McCracken [12], Chow and Hale [9], Guckenheimer and Holmes, Arnold [4], Wiggins [17], Arrowsmith and Place [5], Dumortier et al. [10], Arn′old et al. [3]. These books have a variety of different strategies to cover the subject, and several could be described as blockbusters in the subject! *Elements of applied bifurcation* by Kuznetsov is a very worthy addition to this collection. The author's strategy is very clear: he wishes to make the catalogued bifurcational behaviour as complete as possible and detectable for the nonspecialist by algorithmic methods. The text is set at the interface between undergraduate and postgraduate studies. The former could benefit from carefully selected parts of the text, but, as with most books, it contains far more than is appropriate for a typical semester postgraduate course.

Kuznetsov is careful to introduce the basic ideas of a dynamical system, seen as an evolution operator ϕ_t acting in some state space X. The time t can be either discrete or continuous. The introduction has pace with a discussion of Cantor-type invariant sets, i.e., the Smale horseshoe, being considered on page 12. The danger of moving at this speed early on in the book reminds the reviewer of seeing the saddle-node bifurcation being discussed within pages of the start of the text by Guckenheimer and Holmes [11]. For the expert reading such a text, already aware of the background detail, the temptation is to marvel at the fluidity of discussion and spread of material covered. To the student, the speed of the introduction can be bewildering. One good feature of the first chapter is the appendix on partial differential equations. The temptation is always to split ODEs and PDEs. The solution of a PDE as an evolution operator on a function space is illustrated early on in the book.

The book eventually moves to catalogue bifurcations, but first the foundations are carefully laid: the key role of structural stability in bifurcation is developed. Structural stability requires equivalence of sufficiently nearby systems, and it is the failure of structural stability as parameters are varied in a family of sytems that gives rise to bifurcations. In the halcyon days of the 1960s structural stability was thought to hold the key to understanding the generic behaviour of all systems on a given space. That this was true for continuous time systems on compact 2-manifolds was shown by Peixoto [5]. Moreover, topologically, they could be precisely characterised. The hope of extending this result to higher-dimensional manifolds was quickly cut short when open sets of nonstructurally stable systems on the 3-torus were found by Smale. The emphasis quickly moved to the breakdown of structural stability, and the subject of bifurcation was reborn on the back of the earlier catalogues of planar dynamic behaviour, for example, by Andronov [1]; the burgeoning theory of nonlinear oscillations; and the new global differentiable dynamical systems of Smale [15].

There is a natural ordering to the importance of bifurcations, and this is measured by codimension. Essentially, the more degenerate the nonstructurally stable element in the family, the more parameters are needed to unfold fully all the possible nearby behaviour. The minimum number of parameters required is called the codimension. The more degenerate equilibria are less observable, and the most important are those of lower codimension. Thus there are two chapters on codimension-one bifurcations for the continuous and discrete time cases.

The extension to n-dimensional systems is also considered. The loss of structural stability often occurs on a submanifold, called the centre-manifold. While the system may be topologically changing its structure on the centre-manifold, the behaviour in the normal direction may be persistently attracting, repelling, or saddle-like. It is in this way that bifurcational behaviour in low-dimensional systems is very relevant to, and sometimes literally a part of, higher-dimensional systems.

One of the most important contributions of the book is the process of bringing the developments in bifurcation theory over recent years into good order. This is most clearly manifest in the theory covered over three of the middle chapters. The results are well interspersed with good examples which help the reader overcome some of the pathologies and conceptual problems of the subject. One is on bifurcations arising from the homoclinic and heteroclinic connections covering, in particular, the Shil′nikov theorems [14] on higher-dimensional connections. The others are concerned with two-parameter bifurcations for continuous- and discrete-time dynamical systems. The almost classic Takens-Bogdanov bifurcation [6, 16] is included as well as less familiar codimension-two bifurcations in $\mathbb{R}^3$. The chapters cover some of the latest results in bifurcation theory and also cover work where the behaviour is still not fully understood, for example, the strong resonance behaviour for generalised Neimark-Sacker-Hopf bifurcations. Kuznetsov discusses the question of the precise way in which two invariant circles of a map generically interact to vanish. This was extensively addressed by Chenciner [8] in the 1980s and has still to be fully clarified, but some of the known behaviour is presented.

All descriptions of bifurcations are helped by diagrams, and the book is in general, well illustrated. However, one criticism is the loose "artistic" interpretation of how unstable and stable manifolds interact in the region of transverse homoclinic and heteroclinic intersections. I believe that even simple sketches should preserve the topological patterns of the "looping" of the manifolds, and several diagrams fail to do so. This is not useful for the comprehension of the readers—they should be aware that there are rules and patterns even in tangled manifolds!

Finally, the author gives the reader a good insight into the numerical methods "which", he notes, "in most cases is the only tool to attack real problems." Numerical procedures are introduced for all the fundamental investigations of equilibria, their linearisations, and their associated insets and outsets, as well as more general invariant manifolds. These techniques are now widespread, and there is a good listing of the software available.

The book is a fine addition to the dynamical systems literature. It is good to see, in our modern rush to quick publication, that we as a mathematical community still have time to bring together, and in such a readable and considered form, the important results on our subject. The author is generous in his reference to other work in the field, with bibliographical details at the end of each chapter. In fact, he has taken care to read the current literature and has taken advantage of the best contributions of other authors to give his book a high quality with good insight. The book is not appropriate if the reader wishes to investigate, say, chaotic dynamics or Hamiltonian systems. However, Kuznetsov's exposition is well focussed and achieves its objectives. It enables us to proceed further in our research on bifurcational analysis with a comprehensive discussion of key bifurcations, and the algorithms to detect them, safely banked for our future reference.

References

1. A. Andronov, E. Leontovich, I. Gordon, and A. Maier, *Theory of bifurcations of dynamical systems on a plane*, Israel Prog. for Sci. Trans., 1973. MR **49:**9345
2. A. Andronov and L. Pontryagin, *Systémes grossieres*, Dokl. Acad. Nauk SSSR **14** (1937), 247–251.
3. V. I. Arn′old, V. Afraimovich, Y. Il′yashenko, and L. Sil′nikov, *Theory of bifurcations*, vol. 5, Encyclopaedia of Mathematical Sciences (V. I. Arn′old, ed.), Springer-Verlag, Berlin and New York, 1994.
4. V. I. Arn′old, *Geometrical methods in the theory of ordinary differential equations*, Springer-Verlag, Berlin and New York, 1983. MR **84d:**58023
5. D. K. Arrowsmith and C. M. Place, *Introduction to dynamical systems*, Cambridge Univ. Press, Cambridge, 1990. MR **91g:**58068
6. R. Bogdanov, *Bifurcations of a limit cycle for a family of vector fields on the plane*, Selecta Math. Soviet **1** (4) (1981), 373–388.
7. G. D. Birkhoff, *Dynamical systems*, Amer. Math. Soc. Colloq. Publ., vol. 9, Amer. Math. Soc., Providence, RI, 1927.
8. A. Chenciner, *Courbes fermes invariantes non normalement hyperboliques au voisinage d'une bifurcation de Hopf dégénérée de diffeomorphismes de* $\mathbb{R}^2$, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), 507–510.
9. S.-N. Chow and J. K. Hale, *Methods of bifurcation theory*, Springer-Verlag, Berlin and New York, 1982. MR **84e:**58019
10. F. Dumortier, R. Roussarie, J. Sotomayor, and H. Zoladek, *Bifurcations of planar vector fields*, Springer-Verlag, Berlin and New York, 1991. MR **93f:**58165
11. J. Guckenheimer and P. J. Holmes, *Nonlinear oscillations, dynamical systems & bifurcations of vector fields*, Springer-Verlag, Berlin and New York, 1983.
12. J. Marsden and M. McCracken, *Hopf bifurcation and its applications*, Springer-Verlag, Berlin and New York, 1976. MR **58:**13209
13. H. Poincaré, *Mémoire sur les courbes definies par les équations differentielles I-IV*, Oeuvre I, Gauthier-Villar, Paris, 1928.
14. L. Sil′nikov, *A case of a denumerable set of periodic motions*, Dokl. Akad. Nauk SSSR **160** (1965), 558–561. MR **30:**3262
15. S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817. MR **37:**3598
16. F. Takens, *Forced oscillations and bifurcations*, Comm. Math. Inst., Rijksununiversiteit Utrecht **2** (1974), 1–111. MR **57:**17720
17. S. Wiggins, *Global bifurcations and chaos*, Springer-Verlag, Berlin and New York, 1988. MR **89m:**58057

David K. Arrowsmith
Queen Mary & Westfield College
University of London
E-mail address: D.K.Arrowsmith@qmw.ac.uk