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The technique of pseudodifferential operators, by H. O. Cordes, Cambridge University Press, 1995, xii +382 pp., \$34.95, ISBN 0-521-28443-0

Pseudodifferential analysis on symmetric cones, by A. Unterberger and H. Upmeier, Studies in Advanced Mathematics, CRC Press, Boca Raton, New York, London, and Tokyo, 1996, iii+216 pp., \$59.95, ISBN 0-8493-7873-7

The advent of the New Quantum Mechanics of Heisenberg, Schrödinger and Dirac in the late 1920's started a revolution in the mathematical outlook of quantum physics. The essence was in the non-commutativity of the quantum world rather than in the Planck discreteness of the Old Quantum Mechanics. Its manifest was the Hermann Weyl book *Gruppentheorie und Quantummechanik* (1928, cf. [We]). In particular it presents the Weyl rule for quantization of classical observables, the functions $f(p,q)$ of $2n$ canonical variables $p=(p_1,\dots,p_n)$, $q=(q^1,\dots,q^n)$ on the phase space $\mathbb{R}^{2n}$. The corresponding quantum observables are operators

$$f(P,Q)=\frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^{2n}} d^n\xi d^n x \widetilde{f}(\xi,x)\exp i(\xi P+xQ)$$

where $P=(P_1,\dots,P_n)$, $Q=(Q^1,\dots,Q^n)$ are unbounded canonical operators of partial differentiation $P_j=\frac{1}{i}\partial/\partial q^j$ (here $i=\sqrt{-1}$) and multiplication $Q^j=q^j\cdot$ on the Hilbert space $\mathcal{L}^2(\mathbb{R}^n)$ of wave functions $\psi(q)$, and $\widetilde{f}(\xi,\eta)$ is the Fourier transform of $f(p,q)$. The exponentials $\exp i(\xi P+xQ)$, $(\xi,x)\in\mathbb{R}^{2n}$, are unitary operators on $\mathcal{L}^2(\mathbb{R}^n)$ parametrized by ξ and η. They define the Schrödinger irreducible unitary representation of the Heisenberg group $\mathbb{H}_n$ generated by $(\xi,\,x)$ and $r\in\mathbb{R}$, with the Weyl commutation relation ($r\in\mathbb{R}$)

$$(\xi,x,r)(\xi',x',r')(\xi,x,r)^{-1}(\xi',x',r')^{-1}=(0,0,r+r'+\xi\cdot x'-\xi'\cdot x)\,.$$

According to Weyl [We, p. 275] the "integral need not be interpreted literally, the essential point being that it represents a linear combination of the simple functions" $\exp i(\xi P+xQ)$. This might suggest that $f(P,Q)$ are unitary representations of some extended group algebra of the Heisenberg group $\mathbb{H}_n$. The Weyl operators $f(P,Q)$ are now known as pseudodifferential operators $f\left(\frac{1}{i}\frac{\partial}{\partial q},q\right)$ (ψdo for short). The term itself was introduced by K. Friedrichs in the early 1960's and actually is quite awkward (e.g. usual partial differential operators are ψdo's this way). Thus Weyl's approach places ψdo's in the larger context of the representation theory of Lie groups and certainly should be related to the geometric quantization of our days, so that the pseudodifferential calculus is an extension of the group algebra operations. However, historically this promising development has been arrested by the powerful opposition in J. von Neumann's *Mathematische Grundlagen der Quantenmechanik* (1932) (English translation [vN]) establishing new foundations for the Quantum Mechanics in the theory of unbounded Hermitian operators. The von Neumann theory was destined to play the same role for partial differential equations as its predecessor, the Hilbert theory of infinite-dimensional symmetric forms, for the integral ones. Nevertheless the ψdo's have been widely used by physicists. They have returned to mathematics in the 1960's, as it were, through

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the back door (the general boundary value problems for elliptic operators) in the pioneering work by A. Calderon and A. Zygmund [CZ], A. Dynin [Dy], J. Kohn and L. Nirenberg [KN], J. Bokobza and A. Unterberger [BU], L. Hörmander [H1] and R. Seeley [Se]. In the past 30 years the field has become an industry, the most important operational technique in the analysis of general partial differential equations with variable coefficients (cf. [Sh], [H2], [Ro], [GS]). From the start the general operational rules (the Leibniz ψdo calculus) have not changed much, but the main progress has been in the aforementioned extension of the Heisenberg group algebra culminating in the Hörmander classes [H2, Ch. XVII]. The calculus itself is a kind of operator calculus, which in the case of partial differential operators with constant coefficients is the classical Fourier-Laplace method.

This leads us directly to Cordes' book. For a long time H. O. Cordes has had a vision of reconciliation of the abstract operator methods and the ψdo techniques. In particular he was among the pioneers in the study of C^*-algebras generated by ψdo's, especially on open manifolds (mainly with cyclindrical and conical ends) (cf. [Co]). The book is based on his courses at UC Berkeley. It is self-contained and, according to Cordes himself, is an old-fashioned course in partial differential equations presented in a modern style. The preface says in a somewhat apologetic way: "But the technique of ψdo's in spite of its endless formalisms (as a rule integrals are always distributional integrals, and infinite series are asymptotically convergent) still provides a strongly simplifying principle once the technique is mastered. Thus our present discussion of this technique is justified."

As a textbook, it does not attempt complete generality. Thus it avoids the Fourier integral operators, and the Cordes bi-graded symbol classes are a rather special case of the general Hörmander symbol classes. After a detailed discussion of the basic ψdo calculus it sketches applications to parabolic and elliptic problems (which are the subject of the earlier book [Co]), but the main thrust is a ψdo C^*-algebra approach to hyperbolic equations. Let $K(t)$ be a family of semi-strictly hyperbolic (scalar) partial differential operators and the evolution operator $U(t)$ be defined by the Cauchy problem $\frac{d}{dt}U(t) + iK(t)U(t) = 0, U(0) = 1$. Then the conjugation of the C^*- algebra of bounded ψdo with the evolution operator $U(t)$ generates a flow on the principal symbol space of the C^*-algebra which may be lifted to a flow on the algebra itself. (This is a version of the Egorov theorem.) The proof is based on a useful characterization of ψdo's with $\mathcal{B}^\infty$-symbols on $\mathcal{L}^2(\mathbb{R}^n)$ as bounded operators for which the conjugation with the Weyl operator family $\exp i(\xi P + xQ)$, $(\xi, x) \in \mathbb{R}^{2n}$, is C^∞-smooth in the uniform operator topology. (This, of course, is an exponentiated version of the Beals' abstract ψdo characterization.) Understandably, the flow does not exist for general hyperbolic systems. The physically interesting case of the ψdo algebra invariant under conjugation with the Dirac evolution operator is a fitting grand finale for the whole book.

All in all, I found the book interesting and appropriate as a text for an introductory graduate course. Certainly it reflects the author's originality and experience in graduate teaching. However, I would rather choose for a graduate course in pseudodifferential analysis the recent elegant and short text by A. Grigis and J. Sjöstrand [GS] and/or the careful and thoughtful exposition by M. Shubin [Sh].

The *Pseudodifferential analysis on symmetric cones* by A. Unterberger and H. Upmeier is a lucid research monograph well suited for a special topics graduate course in ψdo's. Essentially it is an introduction to the impressive research program akin to the Weyl philosophy which A. Unterberger (with the help of J. Unterberger and H. Upmeier) has been vigorously developing since the early 1980's (cf. [Un]).

Let Λ be a symmetric (open) cone in $\mathbb{R}^n$, i.e. a cone which is homogeneous under the group $GL(\Lambda)$ of linear transformations of $\mathbb{R}^n$ leaving Λ invariant and such that

$$\Lambda = \{x \in \mathbb{R}^n : x \cdot y > 0 \text{ for all } y \in \Lambda\}\,.$$

The latter implies, of course, that $\Lambda \neq \mathbb{R}^n$. Classical examples are: positive semiaxis $\mathbb{R}^+$ (then $GL(\mathbb{R}^+) \simeq \mathbb{R}^+$) the future light cone (then $GL(\Lambda) =$ the orthochrone Lorentz group), the cone of positive $n \times n$ hermitian matrices (then $GL(\Lambda) = U(n)$ acts by conjugation). By default Λ is a globally symmetric Riemannian space, i.e., there exists a Riemannian metric g on Λ such that for every $x \in \Lambda$ the central geodesic symmetry S_x of Λ at x is a global isometry. Moreover, $GL(\Lambda)$ becomes the group of isometries of (Λ, g). The authors consider the semi-direct product G of $GL(\Lambda)$ and $\mathbb{R}^n$ with the group multiplication

$$(P_1, b_1)(P_2, b_2) := (P_1 P_2, b_1 + (P_1^t)^{-1} b_2),\ \ P \in GL(\Lambda)\,,\ \ b \in \mathbb{R}^n$$

and its unitary representation on $\mathcal{L}^2(\Lambda)$

$$U(P, b)u(t) = u(P^{-1}t)e^{2\pi i \langle b,t \rangle}\,.$$

In particular

$$U(S_x, \xi - S_x \xi)\,,\ \ x \in \Lambda\,,\ \ \xi \in \mathbb{R}^n\,,$$

are unitary operators on $\mathcal{L}^2(\Lambda)$ and the subgroup generated by the $(S_x, \xi - S_x\xi)$ in G is a normal one. The authors introduce their ψdo algebra on Λ as an appropriate extension of the representation of that group algebra. (In case of $\mathbb{R}^n$, their construction gives exactly the Weyl ψdo's.)

A. Unterberger calls the corresponding ψdo's calculus on Λ the Fuchs calculus because for $\Lambda = \mathbb{R}^+$ the corresponding elliptic ordinary differential operators are exactly of the classical Fuchs type. Incidentally, because of the explicit geometric background of the subject, the book has rich connections with classical analysis. Thus the Fuchs calculus is a contraction (as a real parameter $\lambda \to \infty$) of the similar λ-Weyl calculi on the λ-Bergman spaces $\mathcal{H}^2_\lambda(\Lambda + i\mathbb{R}^n)$ of holomorphic functions on the "right half-plane" $\Lambda + i\mathbb{R}^n$. The corresponding group algebras are generated by the holomorphic discrete series of the unitary representations of the holomorphic automorphisms of that "right half-plane". It is remarkable how the Fuchs calculus is similar to the standard Weyl calculus on $\mathbb{R}^n$ including the explicit composition rule and the analogues of the Beals and Egorov theorems, as well as of the action of the diffeomorphisms of Λ. The main techniques are the generalization of the coherent state approach to ψdo's (the Berezin anti-Wick quantization) using the Jordan algebra approach to the geometry and analysis on the symmetric cones (cf. [Up]). Though the applications of the book are only on the horizon, its results should be relevant for ψdo analysis of the boundary value problems on the manifolds with singular boundaries which have conical tips. Then the Fuchs ψdo algebra can be useful when the partial differential operators degenerate completely at such boundaries. This relates both books under review, since the Cordes ψdo's degenerate completely at the infinity on the conical ends.

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