BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 3, July 1996

Geometric group theory, Vol. 2: *Asymptotic invariants of infinite groups*, by M. Gromov, edited by A. Niblo and Martin A. Roller, London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993, vii + 295 pp., \$34.95, ISBN 0-521-44680-5

The graphic work of M. C. Escher is much appreciated by mathematicians and is also enjoyed by a much wider public. A quick look at Escher's pictures of tesselations of the Euclidean and hyperbolic planes makes everyone see that these two geometries have different asymptotic behavior: the Euclidean plane has quadratic polynomial growth, while the hyperbolic plane has exponential growth. The reviewer has found, in teaching courses to audiences not at all inclined to mathematics, that the difference between these two geometries can be immediately understood and appreciated by looking at these pictures.

The evident quadratic and exponential growth that can be seen in these pictures is a special case of one of the earliest and most easily understandable asymptotic invariant of infinite (and finitely generated) groups, namely, the *growth of a group*. Together with the concept of growth came the idea of considering a group to be a *metric space* and to relate this metric space to more conventional metric spaces on which the group acts by the concept of *quasi-isometry*. Namely, if Γ is a finitely generated group and $x \in \Gamma$, define $|x|$, the distance of x from the identity, relative to a set of generators $x_1, \cdots, x_n$ for Γ, to be the minimum length of a word in $x_1^{\pm 1}, \cdots, x_n^{\pm 1}$ expressing the element $x \in \Gamma$. The distance between $x, y \in \Gamma$ is defined to be $|x^{-1}y|$. It is easily checked that this distance makes Γ into a metric space and that different (finite) sets of generators give metrics which are bi-Lipschitz equivalent. The *growth function* of Γ is the function on the natural numbers that to n associates the cardinality of the ball of radius n, i.e., the number of elements in Γ expressible as words of length at most n. The *growth* of Γ is the order of growth of this function.

The original applications of the idea of viewing a group as a metric space can now be formalized in the concept of *quasi-isometry*. Two metric spaces X and Y are called *quasi-isometric* if there exists a map $f : X \to Y$, not necessarily continuous, and constants $C \geq 1$ and $D \geq 0$ so that

$$C^{-1} d_X(x_1, x_2) - D \leq d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2) + D$$

holds for all $x_1, x_2 \in X$, and, given any $y \in Y$, there exists an $x \in X$ such that $d_Y(y, f(x)) < D/2$. One can easily check that any map $g : Y \to X$ defined by assigning to any $y \in Y$ any element $x \in X$ such that $f(x) = y$ satisfies the same conditions for the constants $C, 4CD$, and both compositions fg and gf are within bounded distance of the identity. Thus quasi-isometry is an equivalence relation on metric spaces. Two spaces are quasi-isometric if they "look bi-Lipschitz equivalent from afar" (i.e., where small distances are neglected). The example to keep in mind is X the set of centers of the fundamental domains in an Escher tesselation and Y the whole (Euclidean or hyperbolic) plane of the tesselation. Then f would be the inclusion of X in Y (which in this case would be continuous), while g would be

1991 *Mathematics Subject Classification.* Primary 20F32.

the discontinuous map that to each fundamental domain assigns its center (with choices made in the boundaries). The basic fact underlying both these definitions and what one sees when looking at the Escher tesselations is that a (discrete, finitely generated group) Γ is quasi-isometric to any metric space on which it acts properly and with compact quotient space (identification space). In particular, a finitely presented group is quasi-isometric to the universal cover of any compact Riemannian manifold with the given group as fundamental group. In this last case, the group and the universal covering manifold have the same growth.

The concept of growth of groups and equivalence with coverings was discovered by Svarc in 1955 and re-discovered by Milnor in 1968 (references to the works quoted here can be found in the bibliography of the book under review). The idea of quasi-isometry was implicit there, but it became more explicit, and found deeper and more remarkable applications, in Mostow's proof of his global rigidity theorem for locally symmetric spaces. In particular, the idea of extracting a space from the asymptotic geometry of a group was central to Mostow's work.

The next dramatic development in asymptotic geometry of groups was Gromov's proof of Milnor's conjecture (stated in problem 5603, *American Math. Monthly*, 1968) that groups of polynomial growth are virtually nilpotent (i.e., have a nilpotent subgroup of finite index). Gromov's basic geometric idea was to construct a metric space from the group by generalizing the construction of the real numbers from the integers, in the following way. One can construct $\mathbf{R}$ from $\mathbf{Z}$ by the following process: $\mathbf{R}$ is the limit of the sequence of metric spaces (X_i, d_i), where $X_i = \mathbf{Z}$ with metric d_i being the usual metric divided by i. This has the effect of "dividing" and passing to a limit in a way that makes sense for metric spaces. Gromov proves that for groups of polynomial growth this process gives a limit which is a *locally compact* metric space on which the group acts. The theory of transformation groups developed around the solution of Hilbert's fifth problem (and collected in the book by Montgomery and Zippin) then gives a homomorphism of the group into a Lie group, which leads to a linear representation of the group, and, after much other work, to the desired theorem.

This geometric construction was formulated by van den Dries and Wilkie, in the context of infinitesimals and non-standard real numbers, as what is now called the asymptotic cone. This is a functor from metric spaces to metric spaces that transforms quasi-isometries to bi-Lipschitz maps. This functor can be applied to the metric space resulting from a finite set of generators of a group and gives a bi-Lipschitz class of metric spaces that depends only on the quasi-isometry class of the group. When applied to $\mathbf{Z}^n$ with standard generators, it produces $\mathbf{R}^n$ with the L_1 metric. When applied to a group of polynomial growth, it produces a locally compact metric space. When applied to the fundamental group of a compact manifold of negative curvature, it does not produce anything resembling the universal cover of the original manifold. Instead, it produces a tree which is not locally compact; in fact it is uncountably ramified at each vertex. It is the reviewer's understanding that there is no known functor from groups to spaces that when applied to a surface group produces the hyperbolic plane.

The book under review is an extensive essay by Gromov on the many possible developments and applications of the general ideas just described, as well as of many other sources of asymptotic ideas that cannot be covered within the constraints of a book review (for example, Folner's characterization of amenability). Groups are replaced by the quasi-isometry classes of the metric spaces they define.

Topological and geometric methods are used to study the resulting spaces and thus the asymptotic properties of the original groups. Every conceivable tool is brought in to study the resulting spaces: topology (set-theoretic topology; dimension theory; algebraic topology, including the homotopy theory of spaces with uncountable homotopy groups); ultrafilters and non-standard analysis; Riemannian and allied geometries; as well as the study of intrinsic and extrinsic curvatures, distortion of submanifolds, isoperimetric inequalities, conformal geometry, Carnot geometry; functional analysis methods: L_2 and L_p cohomology, infinite-dimensional representation theory, spectral theory; measure theory in various forms; probability theory; and many other tools. Many potentially interesting asymptotic invariants of groups are introduced, although few are actually studied in depth.

It is most important, in reading this book, to take the author literally in his whole warning paragraph (section 0.4, page 10), including its concluding sentence: "The readers of this paper should not expect new theorems (not even half proved ones), but they may come across some amusing problems." The book is not meant to contain any theorems. It is not a monograph developing a concrete and well-defined new theory, as was the author's famous monograph on hyperbolic groups. It is clear to the reviewer that this book is written solely for the purpose of stimulating thought. The author is trying to convey, in very general terms, his overall vision of the subject. The best way to explain this is that *the book is itself written in the asymptotic spirit*! The author is trying to convey a look of the subject that fits together very well when looked at from afar or in the large scale, but where many details have not yet been worked out (and where in many cases the author has no intention of working them out) and may not yet fit together at all. Indeed, many of the "amusing problems" in the book consist of an invitation (or perhaps a challenge) by the author to the reader to *formulate* and prove precise statements "in the small scale" that the author can only glimpse "asymptotically".

This way of writing is unusual for the mathematical community. We are used to reading what should be polished works, with precise statements and proofs carefully worked out. We also hope that the results we read can be quoted verbatim and used in our own work. This is *not* the nature of this book. The reviewer firmly believes that statements in the literature should never be quoted without further corroboration. But in the case of this book, it is particularly important that statements not be blindly quoted by other authors. This would be as meaningless as trying to make statements about local properties of spaces where one only knows the quasi-isometry class!

In summary, this book is really a collection of problems, presented in an unconventional way. The problems are built from a large collection of instructive and stimulating examples and are held together by a remarkable asymptotic view of the subject. Some problems may have a better-defined direction than others. For example, the general notions of asymptotic cones have been carefully defined in the literature, and there are now clearly formulated problems that are the subject of current research. It is also being used as a tool by some geometers to study other problems. The concept of distortion of subgroups is clearly a useful one that will find applications. The reviewer finds the section on filling invariants of nilpotent groups stimulating, but has to admit that he has been unable to make precise sense of it. But others are working, with success, in this direction.

The number of problems in the book is immense and impossible to review in a meaningful way. Given this situation, it seems best just to mention the reviewer's favorite problem, which is hidden on page 23: Is every separated net in $\mathbf{R}^2$ bi-Lipschitz to $\mathbf{Z}^2$? This is a problem of striking simplicity that lies at the foundation of the whole subject and which has remained unsolved for many years.

The reviewer can recommend to any mathematician picking up this book, glancing through it until he or she finds an area of mathematics dear to his or her heart, then reading that part to see if it makes sense. Once it makes some sense, read it again to see if it generates any new ideas.

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