

Combinatorial rigidity, by Jack Graver, Brigitte Servatius, and Herman Servatius,
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It has been a mystery to me why more thought has not been devoted to the geometry of structures. One could argue that there are areas of mathematics that deal with the “structure of things”, such as the theory of elasticity, distance geometry, the smooth theory of rigid surfaces, for example. But in my mind these subjects are either appendages to other more general interests or they have been sidetracked into irrelevant and uninteresting directions.

On the other hand, there have been some admirable accomplishments coming from attempts to understand rigid structures. One of the first results of A. L. Cauchy in 1813 in [2] was a surprisingly insightful theorem about the rigidity of convex polyhedral surfaces. (Never mind that a mistake went unnoticed and unrepaired for more than 50 years.) After Cauchy the subject that one might call the geometry of discrete rigid structures progressed slowly in the mathematical world. But in the engineering world there was a very healthy and vigorous interest in rigid structures. Bridges, buildings, mechanical gadgets and countless other “things” had to be built. Indeed some interesting mathematical ideas have a background from this sort of engineering. For example, for many years in the nineteenth century the primary method of computing forces on a structure was through what is called “graphical statics”. (See Crapo and Whiteley [5] for a discussion of this classical subject.) Today some of these ideas are finding their way into the mathematical literature. For example, it is not too much of a stretch to say that the “regular or coherent triangulations” of Billera and Sturmfels [1] are extensions of some of the ideas that are involved with graphical statics. However, the lack of an overriding mathematical point of view in the engineering literature has created an underlying subliminal confusion. (See Grünbaum and Shephard’s *Lectures on lost mathematics* [6] for more evidence of this confusion.) Only in the last 25 years or so have proper definitions been put forward to clarify vague ideas and some of the “results” in the engineering literature been sorted into theorems and conjectures.

There is a basic and important question. Is a given structure rigid or is it not? One must decide what the structure is and what it means to be rigid. A natural candidate for such a structure consists of a finite collection of points, a configuration $p = (p_1, \dots, p_n)$, $p_i \in \mathbb{R}^d$ Euclidean space, together with a graph G whose vertices correspond to the points of the configuration and whose edges correspond to pairs of points that are constrained to stay the same distance apart. The graph G and the configuration p , together denoted as $G(p)$, are called a *framework* and the edges of G are called *bars*. The framework $G(p)$ is *rigid* if the only continuous motion of the points of the configuration p maintaining the bar constraints is one coming from a family of congruences (motions of all of Euclidean space that preserve all distances).

The subject naturally develops in two related directions, the geometric and combinatorial. Roughly speaking, the geometric considerations are those that have to

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do with the configuration p , and the combinatorial considerations are those that have to do with the graph G . Cauchy's Theorem is an example of the geometric theory because of the convexity assumption. (For a discussion of rigidity from the point of view of convexity, see Connelly [3].) One says the graph G is (generically) d -rigid if for almost all (an open dense set of) configurations p , the framework $G(p)$ is rigid in \mathbb{R}^d . Combinatorial rigidity theory is concerned with the generic rigidity of such graphs.

From either point of view a natural first step is to "linearize" the problem. Regard the graph G as determining a function, the rigidity map, $f : \mathbb{R}^{dn} \rightarrow \mathbb{R}^e$, where \mathbb{R}^{dn} is the space of all configurations of n points in \mathbb{R}^d , \mathbb{R}^e is the target Euclidean space for the squared edge lengths, and e is the number of edges in G . For a configuration p , the appropriate coordinate of $f(p)$ is the square of the distance between the vertices of the corresponding edge of G . The matrix of the differential df with respect to the natural coordinates \mathbb{R}^{dn} and \mathbb{R}^e is called the *rigidity matrix* $R(p)$ for $G(p)$. The framework $G(p)$ is called *infinitesimally rigid* if the rigidity matrix $R(p)$ is of maximal rank. A basic theorem says that if the framework $G(p)$ is infinitesimally rigid, then it is rigid. From this it is possible to see that the graph G is generically rigid if for some configuration p , the framework $G(p)$ is infinitesimally rigid.

One of the first non-trivial results of the more recent combinatorial theory of rigid graphs is a result of Laman in [8]. Suppose that a graph G has exactly $2n - 3$ edges, where n is the number of vertices of G . Then G is generically rigid in \mathbb{R}^2 if and only if $e' \leq 2n' - 3$ for every subgraph of G with n' vertices and e' edges. This is a completely combinatorial characterization of (generically) rigid graphs in the plane, an example of a success for the combinatorial theory. It is an outstanding problem to find an analogous result for (generically) rigid graphs in three-space. One advantage of such a combinatorial characterization is that because of Laman's Theorem, there is an algorithm to test the (generic) rigidity of any graph in the plane whose running time is proportional to n^2 , where n is the number of vertices of the graph G . See Lovasz, Yemini [9] as well as Hendrickson [7] and Crapo [4] for a discussion (and some improvements) of such algorithms.

Combinatorial rigidity is a much-needed graduate-level introduction to the theory of generically rigid graphs. After a pleasant introduction to the general theory and a brief history in Chapter 1, Chapter 2 discusses infinitesimal rigidity. Chapter 3 continues with a systematic, but somewhat tedious, discussion of matroid theory, especially as it applies to the rigidity theory. Chapter 4 covers the theory for planar rigidity, and Chapter 5 discusses the situation in higher dimensions as well as some of the most relevant unsolved problems. There is also a very useful annotated list of references.

One of the interesting approaches in this book can be exemplified as follows. Regard the configuration p in \mathbb{R}^d as any set of generic points. All the coordinates of p are algebraically independent over the rational numbers. Consider the graph G as the complete graph with all $n(n-1)/2$ possible edges present. Consider the rigidity matrix $R(p)$ of this graph. Each subset of the rows of $R(p)$ will either be independent or not. For any graph on that set of vertices, such information can be used to determine infinitesimal and thus generic rigidity for the corresponding subgraph. This is also the classic example of a matroid. Roughly speaking, a *matroid* is a finite set (for example, the rows of a matrix) together with something generalizing a concept from linear algebra (for example, linear independence and dependence of

any subset of the rows of a matrix) that satisfies a natural set of properties for that concept. *Combinatorial rigidity* then includes additional properties for an abstract matroid that define what the authors call an *abstract rigidity matroid*. So in this context, Laman's Theorem provides a very good combinatorial characterization of this generic rigidity matroid. There are also other examples of abstract matroids, for instance the spline matroids in Whiteley [11].

The only thing that seems to be missing from the book is a discussion of vertex splitting and related subjects. This is an operation first used by W. Whiteley in [10] that takes a generically rigid graph with a minimal number of bars (this is called *isostatic*) and replaces it with another that has one of the vertices replaced by two new ones joined to the old vertices and each other appropriately. The resulting graph is also isostatic. I regard vertex splitting and Laman's Theorem as the two most important ideas in combinatorial rigidity in the last 50 years.

There are a very large number of exercises ranging from the finish-this-boring-proof to the very stimulating and interesting to the unsolved problems of the subject. Generally the book is mistake-free, although there are some annoying but minor mistakes. For a graduate student or for a motivated upper-level undergraduate student, this book is a good, accessible first place to get started learning about the generic rigidity of frameworks.

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