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Applied functional analysis (*Applications to mathematical physics*), by Eberhard Zeidler, Appl. Math. Sci., vol. 108, Springer-Verlag, New York, 1995, xxix + 479 pp., $65.00, ISBN 0-387-94442-7

Functional analysis is primarily concerned with infinite-dimensional spaces, mainly function spaces—topological vector spaces whose "points" are functions—and mappings between them, usually called *operators* or, if the range is on the real line or in the complex plane, *functionals*. Its beginning dates back to 1887, when Volterra published notes in special classes of functionals; but as a large field of its own, it is one of the great mathematical creations of our century.

Volterra used concepts from the calculus of variations, which, together with integral equations, supplied motivations, ideas, techniques, and applications during the early period of the development. Its former name, *functional calculus* (*Calcul fonctionnel*), coined by Hadamard and Fréchet, indicates that the original purpose was the extension of the calculus to the study of functionals. Cf. [3].

During the nineteenth century, problems and methods on spectral theory of ordinary and partial differential equations, potential theory, Fourier expansions, and special functions had been slowly accumulating. Under the influence of physics the study of "general solutions" of functional equations was gradually superseded by that of solutions satisfying additional conditions. The notion of an *equation* as a fundamental concept was more and more replaced by that of an *operator* or *functional*, and various kinds of convergence gained practical significance, a novel feature without counterpart in $\mathbb{R}^n$.

Between 1900 and 1910 there occurred a sudden crystallization of these ideas in Fredholm's work on integral equations (1900, 1903), Lebesgue's thesis of 1902 on integration, Fréchet's thesis of 1906 involving metric space, Hilbert's *Vierte Mitteilung* of the same year on spectral theory and Hilbert sequence space ℓ^2, and F. Riesz's (obscurely published) paper of 1907 on topological space, axiomatically defined with *accumulation point* as the basic concept (at that time called *Verdichtungsstelle*, which did *not* mean condensation point!).

Functional analysis gained momentum when topology had been developed sufficiently far, notably in Hausdorff's very influential *Mengenlehre* [6] of 1914, and became a field of interplay between algebra, topology, and analysis, but mostly with emphasis on the latter and on the *applicability* of major results. Within a decade (1920–30) the fundamental theory of normed spaces was completed (by Banach, F. Riesz, Hahn, Helly, Steinhaus, and others). In 1932 three basic monographs by Banach [2], M. Stone [12], and von Neumann [13] showed that functional analysis had been established as a field of its own. Subsequently, Bourbaki [4] and others developed *topological vector* spaces (topological spaces such that the two algebraic operations of vector space are continuous in the topology) and *locally convex topological vector spaces* (topological vector spaces that have a base of neighborhoods of 0 consisting of convex sets). The latter turned out to be practically the most general topological vector spaces suitable for the majority of the tasks in applied

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functional analysis, arising from ordinary and partial differential equations, integral equations, optimization, Fourier analysis, fluid and heat flows, elasticity theory, quantum mechanics and field theory, numerical analysis, and so on. In the further development of these areas, one can observe a frequent exchange of ideas between theory and practical problems.

The book under review is the first volume of a two-volume *Applied functional analysis*. It contains *Applications to mathematical physics* (the subtitle of the volume). Intended for undergraduate or first-year graduate students, it presupposes only familiarity with calculus (but the Lebesgue integral, Fatou's lemma, etc., are used in Chapter 2). Although the presentation is relatively detailed, it requires a certain degree of mathematical maturity, because on its way to applications it often cannot avoid conceptual or methodical heterogeneity.

The author has chosen an "application-oriented" shortest path to tangible practical results, and he has decided to distribute the material to two volumes (perhaps also for economical reasons—the book is expensive for a student). The present first volume contains various mathematical and physical applications. The second volume is somewhat more advanced and places stronger emphasis on general principles of functional analysis; it includes further applications, mainly to physics, but also to some other fields. The present volume is practically self-contained with respect to most *mathematical* derivations, whereas at a few places further details of *applications* are postponed to Vol. II, so that in such a case it would be better to have Vol. II side-by-side.

The division into two volumes necessitated some deviation from the traditional order of material followed in other books on functional analysis, a majority of which also consider applications, albeit not in such abundance within the space available (cf. [1, 5, 9, 10, 14]). In particular, the four "big" theorems on normed and Banach spaces—the Hahn-Banach, uniform boundedness, open mapping, and closed graph theorems—have been moved to Vol. II.

Nevertheless, the present Chapter 1 on Banach spaces is rich in content. It introduces basic concepts that are needed throughout the book and centers about the fixed point theorems of Banach and of Schauder and their application to ordinary differential equations (Picard's and Peano's theorems) and (nonlinear, Fredholm) integral equations. The important idea of obtaining existence theorems via a-priori estimates is sketched in connection with the Leray-Schauder principle. The elements of spectral theory are explained in the context of Banach algebras, which are introduced *ab ovo*. The chapter closes with linear differential equations in Banach spaces and a short section on the (classical) Weierstrass approximation theorem in connection with density arguments.

Since the present "application-oriented" approach imposes a topical order different from the usual one, it seems advisable to outline the further content in somewhat greater detail.

Hilbert spaces are discussed in Chapters 2–4, with Chapter 2 on the usual elementary theory (inner product, orthogonality, Schwarz inequality, etc.), Chapter 3 on Fourier series and transforms in Hilbert space theory, and Chapter 4 on compact symmetric linear operators. This includes the introduction of Lebesgue space $L^2(G)$, $G \subset \mathbb{R}^n$, and space $C_0^\infty(G) \subset L^2(G)$ of test functions (arbitrarily often partially differentiable functions with compact support in G), and the treatment of the Dirichlet principle, beginning with an existence and uniqueness theorem for quadratic variational problems and leading to an existence theorem for the (generalized) Dirichlet problem for the Poisson equation, via the Euler-Lagrange equation, generalized derivatives, and Sobolev spaces $W_2^1(G)$ and $\overset{\circ}{W}{}_2^1(G)$. Further applications concern the solution of the variational and generalized variational problems corresponding to $u'' = f(x), x \in [a,b], u(a) = u(b) = 0$, as well as an application of finite elements and Ritz's method, with an interpretation in terms of elasticity theory. The chapter closes with the projection theorem, the Riesz representation for continuous linear functionals, the Lax-Milgram theorem, and Friedrichs mollification.

Fourier series, orthogonal expansions, the Gram-Schmidt process with applications to Legendre and Hermite polynomials, and a few paragraphs on unitary operators and the Fourier transform of tempered distributions make up the content of the short Chapter 3. The even shorter Chapter 4 is devoted to the spectral theory of compact symmetric linear operators, with application to a Fredholm integral equation and a problem $-u''(x) = \mu u(x)$ on $(0,\pi), u(0) = u(\pi) = 0$.

Chapter 5, the most extensive and important chapter of the book, impressively demonstrates the close relation of (classical and modern) physics to functional analysis. Mathematically, it centers around self-adjoint operators, as introduced by John von Neumann, and the (self-adjoint) Friedrichs extension of symmetric operators, one of the basic instruments in the modern theory of partial differential equations. Problems studied in this chapter, all abstractly in the Hilbert space setting, are the boundary value problem $Au = f$, Dirichlet variational problem $\frac{1}{2}(Au|u) - f(u) = \min$, boundary-eigenvalue problem $Au - \mu u = f (\mu \in \mathbb{R})$, heat equation $u'(t) + Au(t) = 0$, $(t \geq 0)$, $u(0) = u_0$, wave equation $u''(t) + Au(t) = 0$, $(t \in \mathbb{R})$, Schrödinger equation $u(t) = -iAu(t)$, $(t \in \mathbb{R})$, $u(0) = u_0$, corresponding semigroups as most important instruments for the functional-analytic modeling of physical evolution processes, and a unitary group corresponding to the Schrödinger equation.

Still in Chapter 5, this is followed by applications of Hilbert space theory to quantum mechanics, quantum statistics (Bose-Einstein, Fermi-Dirac), and Fock spaces (for bosons and fermions). Assuming no physical prerequisites, this begins with the basic notions of state, observable, Hamiltonian, mean and variance (here called dispersion), etc., and leading up to spectral representations of self-adjoint operators, the abstract Schrödinger equation, the Heisenberg relation, and C^*-algebras. This detailed and enjoyable chapter concludes with discussions of Dirac's calculus and Feynman's integral—intended to help the mathematician in understanding the thoughts of the physicist—and solitons (Korteweg-de Vries equation) and inverse scattering.

This main chapter is well balanced. More important than the presence or absence of some topic is the general intention of the presentation to bridge the conceptual, notational—and perhaps psychological—gap between physics and mathematics, to the benefit of both fields.

Each chapter is followed by some problems intended to help the reader better understand the text or to extend the material contained in the chapter. Other valuable features are the excellent selection of references and the thorough list of symbols.

Since the work is a textbook, probably for a first course in functional analysis, we add a few marginal comments, which would not be of great importance in the case of a research monograph.

Would it help the beginner to give a motivation for using

$$(u|av + bw) = a(u|v) + b(u|w)$$

instead of the usual $\overline{a}$ and $\overline{b}$ on the right (and similarly on p. 103 the deviation from the usual in the inner product in L^2 and in $\mathbb{C}^n$), which the student may find in other books? Would it be good to say in what way the Peano existence theorem for ODE's generalizes Picard-Lindelöf? And that one loses uniqueness? Of course, the formulas tell it, but will the student see it? Is it wise to use the symbol $C[a, b]$ also in connection with L^2? Would it help the student in orientation in other books if important names (e.g., Fredholm equation, on p. 23) were given?

The author distinguishes the "application oriented way" in his book from "the other way of teaching mathematics, the systematic way." It seems that in practice, one might have to compromise between these two ways, depending on the goal of the course, its role in the total curriculum, and the interest of the students. Moreover, the "systematic way" does not preclude the insertion of motivating applications and examples at any instant of the discussion, from physics, engineering, or other fields. The whole issue of approach would perhaps need further discussion and comparisons of ideas with the reality and practice of teaching goals, also in the light of the present teaching reform that is in progress. Of course, it is of great merit that the author has used his great experience and knowledge in collecting and preparing an unusual number and variety of stimulating physical applications that extend from classical physics to contemporary physical research.

Zeidler's monumental five-volume book [15] has been well received as an encyclopedic work on *nonlinear* functional analysis that has been compared to the famous three volumes of the Dunford-Schwartz monograph [5] on *linear* functional analysis. The present textbook is a valuable addition to the very extensive number of texts on functional analysis of various levels. Its special approach and its rich and inspiring collection of applications will secure the book a good place in the existing literature.

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