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Cardinal arithmetic, by Saharon Shelah, Oxford Logic Guides, vol. 29, Oxford Univ. Press, London and New York, 1994, xxxi + 481 pp., $105.00, ISBN 0-19-853785-9

Set theory has made tremendous progress in the last 75 years, but much of it has been outside the boundaries of the usual axioms, the Zermelo-Fraenkel axioms with the Axiom of Choice, or ZFC. A lot has been connected with the continuum problem, i.e., the question whether $2^{\aleph_0} = \aleph_1$. This was attacked by both Cantor and Hilbert and was mentioned as Problem 1 on Hilbert's famous problem list.

Ultimately the continuum problem was shown to be undecidable in ZFC with quite different methods. Gödel developed the theory of constructible sets, which has metamorphosed into the study of inner models of ZFC (in which additional axioms are usually critical), and Cohen developed the famous theory of forcing and generic sets, an elegant combinatorial way to determine when a proposition is consistent with ZFC (and only indirectly whether a proposition follows from ZFC, in which case a direct proof usually needs to be supplied).

Both these approaches continue unabated nowadays, and both necessarily involve studying propositions presumably consistent with ZFC, like the existence of various sorts of large cardinals.

It sometimes seems that the role of set theory as the study of the consequences of ZFC has been lost, or largely minimized.

Just a few years ago, however, Saharon Shelah found an extensive, powerful, and very beautiful theory entirely within ZFC, utterly unsuspected by anyone. The book under review, *Cardinal arithmetic*, is in part an account of this theory.

One approach to the theory is by way of cardinal exponentiation. Cohen showed by forcing that $2^{\aleph_0}$ could be any of a vast class of cardinals, and this was extended by Easton to show that the same is true with $\aleph_0$ replaced by any regular cardinal.

But singular cardinals, like $\aleph_\omega$, are different. For a long time it was expected that new forcing techniques would be developed to show that $2^{\aleph_\omega}$ could be large in exactly the same way as $2^{\aleph_0}$, but Silver and Shelah showed otherwise. Silver showed that the Generalized Continuum Hypothesis cannot fail for the first time at a singular cardinal like $\aleph_{\omega_1}$, and Shelah showed that the same sorts of restrictions apply to cardinals of countable cofinality (like $\aleph_\omega$) as well.

More recently, Shelah has discovered that the wrong sorts of questions were being asked. What is more important to compute than $2^{\aleph_\omega}$ is $\aleph_\omega^{\aleph_0}$ and other quantities. For example, ${\aleph_\omega}^{\aleph_0}$ is the product of $2^{\aleph_0}$ and the minimal cardinality of a cofinal subset of $(\{\, X \subseteq \aleph_\omega : |X| = \aleph_0 \,\}, \subseteq)$. Now $2^{\aleph_0}$ can vary wildly, but absolute upper bounds can be put on $\mathrm{cf}(\{\, X \subseteq \aleph_\omega : |X| = \aleph_0 \,\}, \subseteq)$ within ZFC itself.

One of the most spectacular applications of the theory is the fact (proved in ZFC!) that if $\aleph_\omega$ is a strong limit cardinal (i.e., $2^{\aleph_n} < \aleph_\omega$ for all n), then $2^{\aleph_\omega} < \aleph_{\omega_4}$. This explains why §2 of Chapter IX is entitled "Why the HELL Is It Four?"

In general, given a set A of regular cardinals, the important set to study is $\prod A$, and even what is called $\prod A/D$, where D is an ultrafilter on A. This consists of

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all functions f on A such that $f(a) < a$ for $a \in A$, where we put $f <_D g$ iff $\{ a : f(a) < g(a) \} \in D$. A quantity of great interest is the cofinality of $<_D$. Similar definitions may be made for ideals I on A in place of ultrafilters on A.

One name for the theory Shelah investigates is pcf theory, the theory of possible cofinalitites, i.e., the theory of $\mathrm{pcf}(A) = \{ \mathrm{cf}(\prod A/D) : D \text{ an ultrafilter on } A \}$ for various A. For example, it is provable that $(\{ X \subseteq \aleph_\omega : |X| = \aleph_0 \}, \subseteq) = \max \mathrm{pcf}(A)$, where $A = \{ \aleph_n : n \geq 1 \}$.

If $\lambda \in \mathrm{pcf}(A)$, then we let $J_{<\lambda} = \{ X \subseteq A :$ for any ultrafilter D on A, if $X \in D$, then $\mathrm{cf} \prod A/D < \lambda \}$. This turns out to be an extremely natural ideal. If we define $f < g \bmod J_{<\lambda}$ for $f, g \in \prod A$ by setting $f < g$ iff $\{ a \in A : g(a) \leq f(a) \} \in J_{<\lambda}$, then this ordering turns out to be λ-directed, and many sequences of functions increasing mod $J_{<\lambda}$ have exact upper bounds.

Such results have been found piecemeal over the past few years under various assumptions. There have appeared several papers attempting to explain this material in an accessible way, but the presentations have often been overtaken by the proof of new results. For example, Burke and Magidor published [1] a well-written account that achieved considerable popularity, and Jech [2] presented a more narrowly focused, more recent excellent account. And the papers continue to be written. A potentially very useful account by Menachem Kojman has just appeared in preprint form. And still more are being planned.

Earlier results often used strong assumptions like $2^{|A|} < \min A$ on the set of regular cardinals, but by and large this is not necessary; $|A| < \min A$ will do. A key tool in dealing with the weaker assumptions is the new ideal $I[\lambda]$. If λ is regular and uncountable and $S \subseteq \lambda$, put $S \in I[\lambda]$ iff there is $\langle P_\alpha : \alpha < \lambda \rangle$ and closed unbounded $E \subseteq \lambda$ such that $P_\alpha \subseteq \mathcal{P}(\alpha)$ and $|P_\alpha| < \lambda$ for all α, and for every $\delta \in S \cap E$ there is $c_\delta \subseteq \delta$ closed and cofinal in δ with order type $< \delta$ and $\forall \gamma < \delta$ $c_\delta \cap \gamma \in \bigcup_{\alpha<\delta} P_\alpha$. $I[\lambda]$ is a normal ideal.

The ideal $I[\lambda]$ is mentioned in the book, and results are often proved under the assumption that certain stationary sets lie in $I[\lambda]$, but much more is known now. If κ, λ are regular and $\kappa^+ < \lambda$, then a stationary subset of $S^\lambda_\kappa = \{ \alpha < \lambda : \mathrm{cf}\, \alpha = \kappa \}$ always lies in $I[\lambda]$. This is a principal tool for finding least upper bounds of sequences of functions.

Another important topic is the way $J_{<\lambda^+}$ is generated from $J_{<\lambda}$ when $\lambda \in \mathrm{pcf}(A)$. It turns out that there is a single subset $B_\lambda \subseteq A$ such that $J_{<\lambda^+}$ is generated by $J_{<\lambda} \cup \{B_\lambda\}$. Assumptions about existence of generators are often used in the book that are now known to be outright provable.

Pcf theory forms only part of the material in the book. Most of it is about the consequences of the theory. For example, if $A = \{ \aleph_n : n \geq 1 \}$, then $\lambda = \aleph_{\omega+1} \in \mathrm{pcf}(A)$ so there is an increasing sequence in $\prod A$ of length λ mod $J_{<\lambda}$, which is cofinal in $\prod B_\lambda$ mod $J_{<\lambda}$ (it has "true cofinality" λ). This is a previously unknown combinatorial structure. It is easily used to prove the existence of a Jońsson algebra on λ, a long-open question. Other examples occur in many fields of mathematics besides logic.

It is the applications of pcf theory that form the lasting material of this book. Pcf theory will continue to develop, and more and more accounts of it will be written, but the applications are unlikely to change in fundamental ways. And in any case, the book is of considerable historical interest in seeing how a brand-new theory may still be developed from nothing.

References

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