

*Subgroup lattices of groups*, by Roland Schmidt, Expositions in Math., vol. 14, de Gruyter, 1994, xv+572 pp., \$148.95, ISBN 3-11-011213-2

Lattice theory originated late in the last century along two strands. Many of the structures used in logic have lattices, particularly distributive lattices, associated with them. The work of Boole [4] and Schröder [27] are seminal in this regard.

The other strand, which concentrated more on the connections with algebra, was initiated with Richard Dedekind [6] in his studies of ideals in algebraic number fields. The lattices he studied usually satisfied his modular law which is sometimes referred to as Dedekind's law.

Lattice theory was used as a tool for deriving some of the basic structure theorems for group theory and for algebraic systems in general. In [19] and [20] O. Ore gave a purely lattice theoretic proof of the Krull-Schmidt theorem on the uniqueness of direct decompositions, considerably broadening the scope of this result. (See [18] for a clear lattice theoretic proof.) In [21] and [22] he applied lattices, which he called structures, to group theory. R. Baer, J. von Neumann and others employed techniques from lattice theory and projective geometry in proving theorems about groups and other algebraic structures, and Marshall Hall's book on group theory contained chapters on both lattice theory and projective geometry.

These results led to the hope that lattice theory might prove to be a powerful tool in group theory. In the introduction to his book [28] Suzuki concluded from his theorem that if  $G$  is a simple group, then  $G$  is determined by the lattice of subgroups of  $G \times G$ , that "we might have a possibility to apply lattice theoretical considerations to solve the classification problem of finite simple groups." However this hope was not realized; much more powerful techniques, primarily character theory and "local analysis", were used. Similarly in Abelian group theory Baer's lattice theory techniques are no longer used. (See page 86 of Kaplansky's monograph [13].)<sup>1</sup> So group theory and lattice theory went their separate ways. (For that matter, group theory nowadays has little in common with Abelian group theory.) Group theory had other techniques and lattice theory had its own deep problems to work on, and most of the applications of lattice theory to algebra were in the field of universal algebra.

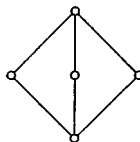
In the last several years some connections between lattice theory and group theory have resurfaced. One problem of interest in general algebra: *is every finite lattice isomorphic to the congruence lattice<sup>2</sup> of some algebraic system?* P. Pálffy and

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<sup>1</sup>A related story: Bjarni Jónsson used lattice theory to define quasi-isomorphism of Abelian groups and to show that, under this notion, the Krull-Schmidt Theorem on the uniqueness of direct decompositions held for torsion-free Abelian groups of finite rank. In the mid-seventies Jónsson bemoaned to me that Fuchs, in the latest version of his book on Abelian groups, had completely removed any trace of lattice theory from his proof of this theorem. A month later I was talking with Lee Lady, who told me that Jónsson did the Abelian group community a great favor by proving his theorem with lattice theory: it forced them to reformulate and reprove it and thereby understand it much better. See Chapter 3 of [15].

<sup>2</sup>If  $f$  is a function with domain  $A$ , then  $f$  determines an equivalence relation on  $A$ . When  $f$  is a homomorphism, this equivalence relation is called a *congruence* relation. The set of all congruence

FIGURE 1.  $M_3$ 

P. Pudlák [24] showed this is closely related to subgroup lattices by reducing the problem to the question: *is every finite lattice isomorphic to an interval sublattice of the lattice of subgroups of a finite group?*

A good test case is the lattice  $M_n$  which is the lattice with  $n$  incomparable elements and a least and greatest element.  $M_3$  is diagrammed in Figure 1. If  $n = q + 1$ , where  $q$  is a prime power, then  $M_n$  is the lattice of subspaces of a vector space  $V$  of dimension two over the field with  $q$  elements and an interval in the lattice of subgroups of the permutation group on  $V$  generated by the translations and scalar multiplication. Can  $M_n$  be an interval in a subgroup lattice of a finite group for the other values of  $n$ ?

Both group theorists and universal algebraists and lattice theorists have worked on this problem. A minimal group whose subgroup lattice has  $M_n$  as an interval has a unique minimal normal subgroup, Köhler [14], and has no Abelian normal subgroup, Pálffy and Pudlak [24], assuming  $n - 1$  is not a prime power. Feit showed that  $M_7$  and  $M_{11}$  do occur as intervals in the subgroup lattice of  $A_{31}$ ; see also Pálffy [23]. The best results are due to A. Lucchini, who has shown that  $M_n$  occurs when  $n = q + 2$  and when  $n = (q^t + 1)/(q + 1) + 1$ , where  $q$  is a prime power and  $t$  is an odd prime [17]. Starting with the O’Nan-Scott Theorem on primitive permutation groups, he and R. Baddeley [1, 16] have reduced the problem of which  $M_n$ ’s can be represented to a list of questions about almost-simple groups. In other words, for  $M_n$  to be representable when  $n$  is not of one of the above forms implies there is an almost-simple group satisfying strong conditions. Although the problem remains open, it appears unlikely that these groups exist for all  $n$ .

Finally we should mention that J. Tuma has shown that every compactly generated lattice, and hence every finite lattice, is isomorphic to an interval in the subgroup lattice of some (infinite) group [29].

A natural question of particular interest to lattice theorists is: *are there identities stronger than the modular law true in all lattices of normal subgroups?* (By a “law” we mean an equation which is identically true.) In [9] it is shown that the Arguesian law, a lattice identity reflecting Desargues law of projective geometry, holds in all lattices of normal subgroups, and in [8] it is shown that there is no finite basis for the laws of the class of all lattices of normal subgroups. P. Pálffy and C. Szabó have exhibited an equation which holds in all subgroup lattices of Abelian groups but does not hold in all lattices of normal subgroups. Their proof [25] is a nice combination of group theory, lattice theory, and projective geometry. It is perhaps difficult to see why this result might be unexpected, but recall that a modular lattice which does not have a sublattice isomorphic to  $M_3$  is distributive and that if  $M_3$  is embedded into the lattice of normal subgroups, then the quotient group of the top of the  $M_3$  over the bottom is Abelian. Thus the “modular” parts of

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relations on  $A$  forms a lattice. If  $A$  is a group, then this lattice is, of course, isomorphic to the lattice of normal subgroups.

the lattice correspond to the Abelian sections of the group. See [5] for a survey of results in this area.

The specific study of the relationship between a group and its lattice of subgroups, the theme of the book under review, began with Ada Rottländer and was championed by R. Baer, who spoke on the subject at the 1938 Symposium on Lattice Theory. A nice body of results was proved early on by R. Baer, K. Iwasawa, O. Ore, E. Sadoval'skiĭ, and M. Suzuki, and Suzuki wrote a monograph on the subject which appeared in 1956. But a great deal of work has been done since then, and this is admirably presented in Schmidt's book. One can get an idea of the amount of work in this area since Suzuki's monograph by comparing the lengths of the books: Schmidt's is 572 pages while Suzuki's is under 100.

The main theme of Schmidt's book is the influence group properties have on the lattices of subgroups and vice versa. For example, which group theoretical properties are determined by the lattice of subgroups? Which groups are determined by their lattice of subgroups and, given a group, which groups have the same lattice of subgroups?

Let  $L(G)$  denote the lattice of subgroups of a group  $G$ . Borrowing the terminology of projective geometry, a lattice isomorphism from  $L(G)$  onto  $L(\overline{G})$  is called a *projectivity* from  $G$  to  $\overline{G}$ .

The first chapter introduces the basic notions and some of the early results proved in this area. Some examples include Ore's theorem that  $L(G)$  is distributive if and only if  $G$  is locally cyclic and its corollary that  $G$  is cyclic if and only if  $L(G)$  is distributive and satisfies the ascending chain condition. Of course this implies that we can identify elements of the lattice of subgroups corresponding to cyclic subgroups.  $G$  is finite if and only if  $L(G)$  is. If  $L$  is a finite lattice which does not have a chain as a direct factor, then there are only finitely many groups having  $L$  as a subgroup lattice.

The second chapter studies groups whose subgroup lattice is modular. These were characterized by K. Iwasawa in the 1940's for locally finite and nonperiodic groups. Schmidt completes the characterization by handling the periodic case. A *Tarski group* is an infinite group, all of whose proper subgroups have prime order. A. Yu. Ol'shanskiĭ [11, 12] showed such groups exist, and Schmidt's characterization is in terms of them. Schmidt also corrects some errors in Iwasawa's proof. Baer's results about projectivities between Abelian groups are also presented in this chapter. Chapter 3 studies groups  $G$  with  $L(G)$  complemented.

Chapters 4, 5, and 6 study projectivities between groups. Chapters 4 and 5 do this for  $G$  finite, while Chapter 6 covers the infinite case. Many important properties of groups are defined in terms of normal subgroups, and so it is important to get a handle on these as much as possible. An element of a lattice is called *modular* if, roughly, the modular law holds whenever it is substituted into one of the variables. Every normal subgroup is a modular element of the subgroup lattice, and more generally every *permutable subgroup* (a subgroup  $M$  with  $MH = HM$  for every subgroup  $H$ ) is a modular element. The image of a normal subgroup under a projectivity is not necessarily normal, while, of course, the image of a modular subgroup is modular.

For a subgroup  $H$  let  $H^G$  be the smallest normal subgroup containing  $H$  (the *normal closure*) and  $H_G$  the largest normal subgroup contained in  $H$  (the *core*).

Structure theorems for  $M^G/M_G$  are given in Chapters 5 and 6 for a modular subgroup  $M$ . If  $\varphi$  is a projectivity from  $G$  to  $\overline{G}$  and  $N \trianglelefteq G$ , let  $H^\varphi$  and  $K^\varphi$  be the normal closure and core of  $N^\varphi$ . Then  $H$  and  $K$  are normal subgroups. A detailed analysis shows that  $H/K$  is solvable of length at most 4 and  $H^\varphi/K^\varphi$  is solvable of length at most 5. This analysis is due to G. Busetto, F. Menegazzo, F. Napolitani, Schmidt, and G. Zacher. These results are used to give lattice theoretic characterization of certain classes of finite groups such as simple groups, perfect groups, solvable groups, and supersolvable groups. More importantly, information about projectivities is obtained.

The situation for infinite groups, studied in Chapter 6, is more complicated. The notion of modular subgroup must be replaced by permodular subgroup. This is a more complicated notion defined in terms of certain subgroups having finite index, and so it is not obvious that it is a pure lattice property. Fortunately it is possible to characterize subgroups of finite index lattice theoretically (and thus the projective image of a subgroup of finite index has finite index). Using this result and the notion of permodularity, one can extend many of the results for finite groups to infinite groups. For example, a finite group  $G$  is simple if and only if  $\{1\}$  and  $G$  are the only modular elements of  $L(G)$ . The Tarski groups mentioned above show that this characterization does not extend to infinite simple groups. Nevertheless a group is simple if and only if  $\{1\}$  and  $G$  are the only permodular subgroups.

Lattice theoretical characterizations are also obtained for perfect, hyperabelian, polycyclic, finitely generated solvable (but not for solvable groups in general), hypercyclic, and supersolvable groups.

A group is *determined* by its subgroup lattice if a projectivity from  $L(G)$  to  $L(\overline{G})$  implies  $G$  is isomorphic to  $\overline{G}$ ; it is *strongly determined* if, in addition, any automorphism of  $L(G)$  is induced from an automorphism of  $G$ . Chapter 7 gives several classes of groups which are determined by their subgroup lattices. For example free groups and non-Abelian torsion-free nilpotent groups are strongly determined. (These results are due to Sadovskii.) Suzuki's result that if  $G$  is a finite simple non-Abelian group,  $G \times G$  is determined is generalized to centerless perfect groups. Using the classification of finite simple groups, one can show that a finite simple non-Abelian group is determined by its lattice of subgroups.

Chapter 8 studies groups for which there is an isomorphism of  $L(G)$  onto the dual of some  $L(\overline{G})$ . Baer began this line of study; he characterized Abelian groups with this property. Zacher showed that a finite group with this property must be solvable and its lattice of subgroups is isomorphic to the lattice of subgroups of an Abelian group. The analysis can be extended to locally finite groups, but the Tarski groups have self-dual lattices, and the structure of arbitrary groups  $G$  with  $L(G)$  isomorphic to the dual of some  $L(\overline{G})$  is not known.

The last chapter studies the lattice of normal subgroups and other lattices associated with groups. Of course the lattice of normal subgroups is much smaller than  $L(G)$  and so tells one much less about  $G$ . (From a general algebra viewpoint it would be more interesting to see what information about  $G$  can be obtained from the normal subgroup lattice of  $K$  for subgroups  $K$  of  $G \times G$  and  $G \times G \times G$ .)

Schmidt is to be commended on a very nice book with a nice collection of interesting results. The presentation is very good. While most of the arguments are group theoretic, they are easily accessible to nonspecialists.

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