BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 4, October 1996

Noncommutative geometry, by Alain Connes, Academic Press, Paris, 1994, xiii+661 pp., ISBN 0-12-185860-X; originally published in French by InterEditions, Paris (*Geometrie Non Commutative*, 1990)

Abstract analysis is one of the youngest branches of mathematics, but is now quite pervasive. However, it was not so many years ago that it was considered rather strange. The generic attitude of mathematicians before World War II can be briefly evoked by the following story told by the late Norman Levinson. During Levinson's postdoctoral year at Cambridge studying with Hardy, von Neumann came through and gave a lecture. Hardy (a quintessential classical analyst) remarked afterwards, "Obviously a very intelligent young man. But was that *mathematics*?"

Abstract analysis was born in the '20s from the challenge of quantum mechanics and the response thereto of von Neumann and M. H. Stone. The Stone-von Neumann theorem (e.g., [1]), which specified the structure of the generic unitary representation of the Weyl relations, established the equivalence of the Heisenberg and Schrödinger quantum mechanical formalisms. (It is interesting that the latter two men seem never to have understood this. The moral may be that rigorous mathematical physicists should not expect the approbation of the practical ones, however fundamental their work.) This was the first nontrivial structure theorem for an infinite-dimensional unitary representation of a noncommutative group and as such an important prototype for infinite-dimensional Lie group representation theory. Von Neumann's writings make it clear that he well understood the importance of this theory for quantum mechanics, and he strongly encouraged his friend Wigner to develop the theory in the case of the Poincaré group. Wigner's paper doing so [2] later became the most referenced work of the twentieth century[1].

Von Neumann himself took on the greater challenge of the mathematics of quantum phenomenology. Together with Stone he established the spectral theorem in complex Hilbert space and the extension theory of hermitian operators, largely a culmination of the work begun by Hilbert, whose prescience two decades in using the term "spectrum" still seems remarkable.

1991 *Mathematics Subject Classification.* Primary 82Exx.

[1] A collection of the most referenced works was published years later by Springer. It is interesting, and representative of the relations between mathematics and physics, that Wigner's paper was originally submitted to a Springer physics journal. It was rejected, and Wigner was seeking a physics journal that might take it when von Neumann told him not to worry, he would get it into the *Annals of Mathematics.* Wigner was happy to accept his offer. (Verbal information from Wigner.)

Soon afterwards, von Neumann proved the double commutor theorem [3], which at the time was a striking theorem in the noncommutative context and thus specifically in the quantum-mechanical direction. The paper was highly suggestive of an infinite-dimensional theory comparable to the Wedderburn structure theorem. In retrospect, it marks the beginning of abstract analysis and led to the classic series of papers on *Rings of operators*, perhaps the most original major work in mathematics in the twentieth century.

There were three principal motivations for the series of papers on "rings", as von Neumann called them. The primary one was, to be sure, the challenge of quantum mechanics. In particular, the divergent character of quantum field theory contrasted oddly with its simple intuitive basis, and von Neumann hoped to reconcile these contrasting features by a suitable formalism. A second significant potential application was to the structure of infinite-dimensional representations of nonabelian groups, which he had already solved in the case of the Heisenberg group. A third motivation was the generalization of the Wedderburn structure theorems.

Today the divergences in quantum field theory appear as probable consequences of an oversimple space-time geometry. In the Einstein universe $R^1 \times S^3$, which is a no less reasonable model for space-time than Minkowski space and locally approximates it arbitrarily closely, the divergences are absent in the crucial case of quantum electrodynamics [4] and in all likelihood in electroweak theory as well. Extensions and applications of the Poincaré lemma to the infinite-dimensional case serve to establish relativistically invariant integrals of powers of quantized fields as selfadjoint operators in Hilbert space [5, 6, 7].

Noncommutative integration theory, in which operators rather than functions are integrated relative to countably additive measures on projections rather than sets [8], together with von Neumann's direct integral (i.e., algebra decomposition) theory, adequately fulfills the basic abstract needs of group theory, e.g., the establishment of the Plancherel theorem for locally compact groups [9]. The Wedderburn structure theory has been extended to arbitrary "type I" rings, i.e., the type that most commonly occurs in practice. The other types are scarcely less refractory than they appeared after the completion of their global theory several decades ago. The local theory dependent on the classification of the central simple rings, or "factors", has been investigated with much ingenuity and energy, but a full classification remains elusive and appears hardly more realizable than a full classification for infinite discrete groups, nor is there any apparent great need for such a classification in other parts of mathematical science.

Thus, von Neumann's basic program has in substantial part effectively been realized. But noncommutative operator algebra is a natural framework for a range of secondary applications, from quantum theory to topology, algebra, and differential geometry. There is little doubt that the author of this handsome, nicely printed book on these applications may be unsurpassed among abstract analysts of his generation. The subject of "noncommutative geometry" has its origin in the von Neumann program for rings of operators together with the representation theory for Banach algebras of Stone, Gelfand, and others. The latter work provided a geometric interpretation for commutative algebras in the form of a maximal ideal or similar space. This pre-war work was quickly followed by the noncommutative extension due to Gelfand and Neumark that brought the Banach algebra theory in close relation to Hilbert space. From a quantum mechanical standpoint, the thrust of these works was that "space-time" was more logically not a primary concept but one derived from the algebra of "observables" (or operators). In its simplest form, the idea was space-time might be, at a deeper level, the spectrum of an appropriate commutative subalgebra that is invariant under the fundamental symmetry group (e.g., [11]).

The algebras investigated by Murray and von Neumann (henceforth, MvN) and those investigated by the Gelfand school were formally similar and in fact essentially identical in the finite-dimensional case. But they differ in topology in the infinite-dimensional case, which at first glance may appear as an essentially technical distinction, but is in fact one that rather fundamentally changes the character of the theory. The algebras of MvN were closed in the weak operator topology, while those of Gelfand were closed in the uniform (operator bound norm) topology. Both are closed under operator adjunction, thus are "*-algebras", hence are appropriately called W^*- and C^*-algebras. The name "von Neumann algebra" was later proposed by Dixmier for a W^*-algebra.

The discovery by Gelfand and Neumark in 1943 of an intrinsic purely algebraic characterization for C^*-algebra was a watershed in the theory operator of *-algebras and has strongly influenced operator-algebraic research. (Von Neumann told me that he was really surprised by it.) In particular, it provides a ready-made noncommutative generalization of the notion of topological space, expecially those that are locally compact. The point is that a locally compact space is essentially equivalent to the algebra of continuous functions thereon, as shown finally by Gelfand and Kolmogoroff, completing a line of investigation by Banach and Stone. This algebra of continuous functions is essentially just a generic *commutative* C^*-algebra. Thus the noncommutative C^*-algebras can reasonably be considered to define a kind of noncommutative "space". Starting from this connection, topological invariants of a space can be reformulated in terms of an algebra of functions thereon and then extended to general C^*-algebras.

At the same time, C^*-algebras were seen to provide a natural framework for quantum phenomenology: the relations of observables, states, spectra, and the general role of group invariance. The attainment of such a framework had been one of von Neumann's central objectives, and in the late '30s he wrote an influential if ultimately inconclusive paper in this direction that was essentially based on the W^*-algebra point of view. The hunch that he indicated to me was that the physicists were "in the wrong ring", i.e., W^*-algebra. (He was particularly fond of the "approximately finite" factor of type II_1, which in fact plays a basic role in the representation of fermion fields.) The Gelfand-Neumark theory provided the missing link to a simple and comprehensive mathematical formulation of quantum phenomenology. In the mid and later '40s, when I undertook to merge the von Neumann and Gelfand outlooks and set up the modern notions of state and observable, this work was thought quite eccentric by many (e.g., [12]). But today it seems hardly more controversial than euclidean geometry.

With a bit more structure, one is led to extend quantum theory from its original Minkowski space-time context to general space-times and to correlate topological invariants with quantum-theoretic concepts, such as Heisenberg canonical systems. Many differential geometric concepts also extend in a natural way to the noncommutative context, e.g., that of differential form. A corresponding noncommutative, or "quantum" notion, arises naturally from the isomorphism between functions and operators that underlies commutative spectral theory. If f is a function on a measure space M, it determines a normal operator M_f on the Hilbert space

$H = L_2(M)$, consisting of multiplication by f, acting on the domain of functions g in H such that fg is also in H. Conversely any normal operator in Hilbert space (or set of commuting normal operators) is unitarily equivalent to such a multiplication operator (or set of such). In addition, the map $f \to M_f$ is evidently an algebraic *-isomorphism, which is essentially applicable to unbounded ("measurable") as well as bounded operators. Thus again there is a natural basis for replacement of functions on spaces by operators.

To give a simple example, if f is a function in euclidean space, its differential $df = \sum_j (\partial_j f) dx_j$ (where $\partial_j = \partial/\partial x_j$) is equivalent to the "differential" of the corresponding multiplication operator, "dM_f" $= \sum_j [\partial_j, M_f] dx_j$. This observation leads to the formulation of the differential dA of an operator A on a given space as the linear map $X \to [X, A]$ from vector fields to operators. A "quantized" 1-form is correspondingly essentially a suitable linear map from vector fields to operators. With suitable restrictions, d extends, along with notions of closure and exactness, to forms of arbitrary degree. Thus quantized forms appear as natural within a purely mathematical framework, but they involve more than the "generalized abstract nonsense" that they may seem to represent. They arose from quantum field theoretic considerations and form an essential ingredient, in the infinite-dimensional case, in a systematic treatment of the "products" of quantized fields, which are at best operator-valued distributions (e.g., [5, 6, 7]).

As indicated above, the difference between C^*- and W^*-algebras is quite functional and not merely technical. Because of the intrinsic algebraic characterization of C^*-algebras, the natural notion of equivalence is that of *-isomorphism. Unitary equivalence of course implies algebraic isomorphism, but the corresponding invariants are too complicated to be very useful. Even under the *-isomorphism equivalence relation, the invariants are not that simple. There are just as many equivalence classes of commutative C^*-algebras as there are locally compact spaces, and the equivalence classes of noncommutative C^*-algebras are vastly more extensive.

In the case of W^*-algebras, unitary (or "spatial") equivalence is the natural notion, and classification is facilitated by the fact that algebraic isomorphism in conjunction with a suitable lack of nontrivial multiplicity suffices to imply unitary equivalence. The simplest case is that of maximal abelian self-adjoint algebras of bounded operators on Hilbert space, for which *-algebraic isomorphism implies unitary equivalence. But in general the theory of W^*-algebras has to do with invariant subspaces and the corresponding reduction of the algebra.

In the case of W^*-algebras of "type I" in the MvN scheme, there is a quite explicit structure theorem [10] that classifies these algebras, within unitary equivalence. The type I algebras include most of the algebras that "really come up" in concrete analytic practice, but a highly notable exception is that of the Clifford algebra over a Hilbert space. This is the simplest of the type II W^*-algebras, and its discovery in the early '30s by von Neumann (in a different form [13]) made it strikingly clear that there were qualitatively new phenomena in infinite-dimensional W^*-algebras. In particular, the Clifford algebra is a central simple algebra that is altogether different from the algebra of all bounded operators on Hilbert space. At the same time, this algebra plays a fundamental role in the analysis of free fermionic quantum fields. In terms of a commonly used notation, it is the algebra generated by the canonical fermionic Q's, and so the analog of the abelian algebra generated by the bosonic canonical Q's. The Clifford algebra is the simplest of the factors that are direct limits of matrix algebras, and as yet only such relatively accessible factor families have been fully classified.

C^*- and W^*-algebras have in common that both support notions of integration. A state E of a C^*-algebra is a positive linear functional, which in the commutative case is equivalent to a measure on the spectrum (or maximal ideal space) of the algebra. In the noncommutative case it is in general not central, i.e., $E(AB)$ and $E(BA)$ are generically unequal. This limits the breadth of C^*-algebraic integration theory, but the plenitude of states makes their limited theory quite useful nevertheless. For example, positive definite functions on groups determine states and can effectively be treated in terms of them. There are generalizations of the Radon-Nikodym and other basic theorems in measure theory to the C^*-algebra context, beginning with [12, 14].

A "weight" on a W^*-algebra A is a countably additive and nonnegative function on projections that is unitarily invariant and thus similarly generalizes the notion of measure applicable to the abelian case. All the basic theorems and ideas of Lebesgue integration theory extend to the noncommutative case, and "measurable" operators can be added and multiplied with abandon, quite like measurable functions and unlike unbounded operators in general. A typical application is the Plancherel theory for general locally compact unimodular groups. When A has a trivial center, i.e., is a "factor", there is an essentially unique weight, which is the dimension function of MvN.

This is the foundation on which *Noncommutative geometry* rests. The "geometry" part of the title is a bit programmatic; a more descriptive title might be *Topics in operator algebra and its applications*. The approach is subjective and informal, although technically highly sophisticated. The author runs through the main features of his own and related works of the past three decades with minimal attention (or references) to earlier, including some ideationally basic, works. The result appears as a brilliant and sustained tour de force that will fascinate sufficiently knowledgeable devotees of the author's main lines of investigation but may baffle nonspecialists. This would be a pity, for the book goes into relations between diverse aspects of its subject that are otherwise not well represented in the literature, especially those in a topological or homological algebraic direction.

Each of the six chapters has its own theme, representing a major interest of the author: I, "Noncommutative spaces and measure theory"; II, "Topology and K-theory"; III, "Cyclic cohomology and differential geometry"; IV, "Quantized calculus"; V. "Operator algebras"; VI, "The metric aspect of noncommutative geometry". All chapters involve *-algebras to a greater or lesser extent but are otherwise only somewhat loosely connected. An introduction provides a prospectus for each chapter. Among the favorite themes that recur are (i) foliations, (ii) cohomology, (iii) K-theory, (iv) the classification of factors, and (v) interpretations of aspects of mathematical physics.

Presently the most novel of these is perhaps the last. It represents an interesting and courageous attempt to bring the author's considerable expertise and flexibility to bear on a variety of topics in physics that he sees as related to operator algebra. These include the KMS (Kubo-Martin-Schwinger) states, whose theory he apparently considers fundamental for statistical mechanics. Unfortunately, there is no apparent empirical basis for this, and at this late date the theory appears as an abstraction from a physical initiative that has not been especially fruitful.

A more lively physical topic is the (two-dimensional) quantum Hall effect, which manifests itself by the integrality (or rationality) of certain observed quantities. A variety of mathematical hypotheses have been advanced in the mathematical physical literature to explain these observations. The book presents a mathematically striking but intricate theory due to Bellisard that deals with the integrality cases. This is based on the "noncommutative torus", an application of cyclic cohomology.

The most remarkable initiative in the physical direction is an interpretation of the unification of the electromagnetic and weak interactions. The "standard model" due to Glashow, Weinberg, and Salam has been empirically quite successful, modulo suitable renormalization prescriptions and extensive radiative corrections. As the author emphasizes, this model is basically phenomenological and is hardly acceptable as an ultimate theory. The book presents a sophisticated mathematical reformulation of the standard model, which it is hoped has the potential to go beyond it. It is exemplary for the author to apply his advanced mathematical expertise to theoretical physics; higher mathematics has become all too introverted and needs the challenge of external connections. However, the interpretation given is basically *descriptive* rather than *explanatory.* No resolution for the long-standing basic question of why the electromagnetic interaction is invariant under space inversion while the weak interactions are maximally noninvariant is proposed. More broadly, the efforts toward issues in mathematical physics seem to ignore some rather basic physical principles—causality, stability (effectively, positivity of the energy), etc.—in favor of a quick technological fix. This appears to represent a kind of naiveté that contrasts oddly with the very high level of mathematical knowledgeability.

Mathematically the most novel chapter of the book is that dealing with cyclic cohomology, which essentially subsumes the quantized forms earlier described but is much more general. It is, however, far from being a systematic exposition of the subject, nor is it intended as such. It is rather an essay on various ramifications, applications to foliations, topology, and the like. It appears technically quite ingenious but perhaps a bit short on independently interesting external issues.

The otherwise most authoritative chapter is that on W^*-algebras, which mainly treats the classification of factors. The early idea of MvN was that the structure of general W^*-algebras might be reducible to that of factors via the decomposition of the algebra with respect to its center. The reduction theory of von Neumann provided a decomposition of any given W^*-algebra (on a separable space) relative to any given Boolean algebra of invariant subspaces. This implies in particular that any W^*-algebra is a "direct integral" of factors. In a formal way, this "local" theory reduces the problem to the case of factors, but the "global" theory is more powerful and illuminating in practice and less fraught with measurability difficulties. Thus, in harmonic analysis over noncompact simple Lie groups the only factor involved is that of all bounded linear operators on a separable Hilbert space, as follows from early work of Harish-Chandra, but this is a long way from establishing the Plancherel theory for such groups.

The book effectively conveys the flavor and spirit of Connes's theory and should be quite stimulating and useful for those doing research in the area. On the other hand, its technical emphasis and the short shrift given to the ideational aspects and historical origins may be offputting to nonspecialists. It would probably be quite difficult to use the book as a reference or source of precise information, somewhat more so because of the lack of page numbers in the index. It is more in the nature of a long discourse or letter to friends. Indeed, its informal style approximates the opposite of the crisp definition-theorem-proof style that one might expect from Connes, and this may be a bit frustrating to those who are not already on top of the material. There is, however, a lengthy bibliography and a brief summary of notation.

As an effervescent and yet sustained account of a wide range of advanced and refined aspects of abstract analysis, this book appears quite valuable and is certainly unique.

References

1. J. von Neumann, *Die Eindeutigket der Schrodingerschen Operatoren*, Math. Ann. **104** (1931), 570–578.
2. E. Wigner, *On unitary representations of the inhomogeneous Lorentz group*, Ann. Math. **40** (1939), 149–204.
3. J. von Neumann, *Zur Algebra der Funktionoperatoren*, Math. Ann. **102** (1929), 370–427.
4. I. Segal and Z. Zhou, *Convergence of quantum electrodynamics in a curved deformation of Minkowski space*, Ann. Phys. **232** (1994), 61–87. MR **95c:**81174
5. J. Pedersen, I. Segal, and Z. Zhou, *Nonlinear quantum fields in ≥ 4 dimensions and cohomology of the infinite Heisenberg group*, Trans. Amer. Math. Soc. **345** (1994), 73–95. MR **95a:**81158
6. I. Segal, *Rigorous covariant form of the correspondence principle*, Proceedings, 1994 J. von Neumann Symposium (W. Arveson, T. Branson, I. Segal, eds.), Amer. Math. Soc., Providence, RI, 1996, 175–202.
7. ———, *Complex noncommutative infinite-dimensional analysis*, Proceedings, 1994 Norbert Wiener Symposium (D. Jerison, I. Singer, and D. Stroock, eds.) (in preparation).
8. ———, *A non-commutative extension of abstract integration*, Ann. Math. (2) **57** (1953), 401–457. MR **14:**991f
9. ———, *An extension of Plancherel's theorem to separable unimodular groups*, Ann. Math. (2) **52** (1950), 272–292. MR **12:**157f
10. ———, *Decompositions of operator algebras*, Mem. Amer. Math. Soc. No. 9 (1951). MR **13:**472b
11. ———, *A class of operator algebras which are determined by groups*, Duke Math. J. **18** (1951), 221–265. MR **13:**534b
12. ———, *Irreducible representations of operator algebras*, Bull. Amer. Math. Soc. **53** (1947), 73–88. MR **8:**520b
13. J. von Neumann, *Continuous geometry*, Proc. Nat. Acad. Sci. U.S.A. **22** (1936), 92–100.
14. H. A. Dye, *The Radon-Nikodym theorem for finite rings of operators*, Trans. Amer. Math. Soc. **72** (1952), 243–280. MR **13:**662b

Irving Segal
Massachusetts Institute of Technology