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SINGULARITIES OF HARMONIC MAPS

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ABSTRACT. This article surveys research on the existence, structure, behavior, and asymptotics of singularities of harmonic maps.

INTRODUCTION

A harmonic map between Riemannian manifolds can be thought of as a natural generalization of a geodesic (where the domain dimension is one) or of a harmonic function (where the range is a Euclidean space). Both of these are smooth objects. However, in general, singularities (that is, discontinuities) do occur in many interesting harmonic maps for reasons of topological obstruction or energy efficiency or as limits of smooth maps minimizing various functionals. In such cases, the location and nature of the singular set and of the mapping nearby may be subject to topological, geometric, and analytic constraints. Recent years have seen many exciting inroads towards understanding these constraints, some of which we will describe here. Thus our focus is *not* on regular harmonic maps and their many applications (see e.g. the surveys [34, 35] and bibliography [13]) but rather on issues involving singularities.

Classically, a real-valued harmonic function h on a bounded domain Ω may be characterized as a critical point of the Dirichlet energy $\int_\Omega |\nabla h|^2 dx$. Similarly, a smooth map u between Riemannian manifolds (M, g) and (N, h) is a critical point for the energy

$$E(u) = \int_M |\nabla u|^2 d \ \mathrm{vol}_M,$$

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where $|\nabla u|^2$ is the square of the length of the differential, which has in local coordinates the form

$$|\nabla u|^2 = \sum_{i,j,\alpha,\beta} g^{ij} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\beta}{\partial x^j} h_{\alpha\beta},$$

and $d\ \mathrm{vol}_M x$ is the volume element, which has the local form

$$d\ \mathrm{vol}_M = (\det g_{ij})^{1/2} dx\ .$$

We will assume that N is isometrically embedded as a subset of $\mathbb{R}^k$. Then, u satisfies the Euler-Lagrange equation

$$-\Delta_M u = A_N(u)(\nabla u, \nabla u) \tag{1}$$

where Δ_M is the Laplace-Beltrami operator of M and $A_N(u)$ is the second fundamental form of N, evaluated at u. For example (1) becomes simply

$$-\Delta u = |\nabla u|^2 u \tag{2}$$

in case M is a domain in $\mathbb{R}^n$ and $N = \mathbb{S}^{k-1} \subset \mathbb{R}^k$. Note also that, for $n \geq 3$, the expression $u(x) = \frac{x}{|x|}$, which exhibits an isolated singularity at 0, has finite energy and defines a distribution solution of (2).

1. The Bochner identity

To understand singularity, let us first recall some criteria for the regularity of a harmonic map. In their pioneering work in 1964 [36], J. Eells and J.H. Sampson established the existence, for any compact M and non-positively curved compact N, of a smooth harmonic map in each homotopy class. A key ingredient in their work is the parabolic version of the following *Bochner identity* for a smooth harmonic map $u : M \to N$ (see [34], §6)

$$\Delta(|\nabla u|^2) = |\nabla(du)|^2 + \langle \mathrm{Ric}_M \nabla u, \nabla u\rangle - \langle \mathrm{Riem}_N(u)(\nabla u, \nabla u)\nabla u, \nabla u\rangle\ . \tag{3}$$

S. Hildebrandt, H. Kaul, and K.O. Widman [69] and others later established results on the existence, uniqueness, and regularity of a weakly harmonic map whose image had small diameter or lies in the domain of a strictly convex function.

2. Weak and stationary harmonic maps

In [96], R. Schoen describes several useful classes of harmonic maps. To describe these, we will employ the space $H^1(M, N)$ of L^2 maps $u : M \to \mathbb{R}^k$ with distribution gradient $\nabla u \in L^2$ and with values $u(x) \in N$ for a.e. x. Most generally, a (*weakly*) *harmonic map* is a locally H^1 weak solution of (1). This is equivalent to assuming that, for any variation $\zeta \in C_0^\infty(M, \mathbb{R}^k)$,

$$\frac{d}{dt}\Big|_{t=0} E(\pi_N(u + t\zeta)) = 0$$

where π_N is the nearest point retraction from some tubular neighborhood of N. A harmonic map is called *stationary* if it is also energy critical for domain variations; that is,

$$\frac{d}{dt}\Big|_{t=0} E(u \circ \Phi_t) = 0$$

where Φ_t, for $|t|$ small, is any smooth deformation of the identity of M such that $\{x : \Phi_t(x) \neq x\}$ has compact closure in M. Any smooth harmonic map, as well as any *energy-minimizing* map, is stationary. An elementary example of a non-stationary harmonic map (of finite energy) is $\omega\left(\frac{x}{|x|}\right) : \mathbb{B}^3 \to \mathbb{S}^2$ where $\omega : \mathbb{S}^2 \to \mathbb{S}^2$ is the stereographic lift of the holomorphic map $2z : \mathbb{C} \to \mathbb{C}$. Here, a non-zero first variation may result by moving the origin in $\mathbb{B}^3$ [12].

For domain dimension $m = 1$, the regularity of a (weakly) harmonic map is immediate from the geodesic ODE. For $m = 2$, the problem is more difficult. Here the regularity was established for energy-minimizers by C.E.B. Morrey [85] in 1948, for conformal harmonic maps by M. Grüter [48] in 1981, for stationary harmonic maps by R. Schoen [96] in 1983 and finally for general weakly harmonic maps by F. Hélein [68] in 1991.

For $m \geq 3$, by contrast, T. Riviere [93] gave an example of an *everywhere discontinuous* weakly harmonic map u from $\mathbb{B}^3$ to $\mathbb{S}^2$. The *partial regularity* of energy-minimizing and stationary harmonic maps will be discussed below.

The stationary assumption allows one to consider radial deformations about a point a in the domain. This leads to the monotonicity of normalized energy. To simplify notations without essentially changing the nature of the regularity and singularity theorems, we will henceforth assume that M is an open domain in $\mathbb{R}^n$. Then the *monotonicity identity* has the simple form

$$\begin{aligned} \sigma^{2-n}\int_{\mathbb{B}_\sigma(a)} |\nabla u|^2 dx - \rho^{2-n}\int_{\mathbb{B}_\rho(a)} |\nabla u|^2 dx \\ = 2\int_{\mathbb{B}_\sigma(a)\setminus\mathbb{B}_\rho(a)} |x|^{n-2}\left(\frac{\partial u}{\partial r}\right)^2 dx \geq 0 \end{aligned} \tag{4}$$

whenever $\mathbb{B}_\rho(a) \subset \mathbb{B}_\sigma(a) \subset M$. The monotonicity inequality along with the *Bochner inequality*

$$\Delta(|\nabla u|^2) \geq -c(n, N)|\nabla u|^2(1 + |\nabla u|^2),$$

which follows from (3), are used by R. Schoen in [96] to prove the interior gradient bound

$$\sup_{\mathbb{B}_{1/2}} |\nabla u|^2 \leq c(n, N)\int_{\mathbb{B}_1} |\nabla u|^2 dx \tag{5}$$

for a *smooth* harmonic map $u : \mathbb{B}_1^n \to N$ having sufficiently small energy. The article [96] also discussed the *higher regularity* of harmonic maps, that a Hölder continuous harmonic map is smooth, depending on M and N, e.g. C^∞ or real analytic if both M and N are C^∞ or real analytic.

3. Partial regularity

In 1983 the important work [97] of R. Schoen and K. Uhlenbeck on the partial regularity of energy-minimizing maps appeared.

Theorem 1. [97] *An energy-minimizing map $u : M \to N$ is smooth away from a closed (singular) subset of M that is discrete in M if $n = \dim M = 3$ and has Hausdorff dimension $\leq n - 3$ in case $n \geq 4$.*

A similar result was obtained by M. Giaquinata and E. Giusti [43] in case the image of u lies in a coordinate neighborhood of N. The paper [97] has many ideas and techniques that have proved to have wide influence in geometric analysis. One of these was the *small energy regularity theorem*, which is analogous to the small excess regularity theorem popularized in geometric measure theory [28, 38].

Theorem 2. [97] *There are positive constants $\epsilon_0 = \epsilon_0(n, N)$, $C = C(n, N)$ so that if $u : \mathbb{B}_1^n \to N$ is an energy-minimizing map with $\epsilon = \int_{\mathbb{B}_1^n} |\nabla u|^2 dx \le \epsilon_0$, then*

$$\sup_{\mathbb{B}_{1/2}} |\nabla u|^2 \le C\epsilon.$$

There are now several proofs of this with different interesting aspects. We will sketch a particularly short one by Y. Chen and F.H. Lin [19] that involves a Ginzburg-Landau approximation. Here one considers a family of functions $w_\delta : \mathbb{B}_1^n \to \mathbb{R}^k$ such that

$$w_\delta = \begin{cases} u & \text{on } \mathbb{B}_{1/2} \\ v_\delta & \text{on } \mathbb{B}_1 \backslash \mathbb{B}_{1/2} \end{cases}$$

where v_δ minimizes the penalized energy $E_\delta(v) = \int_{\mathbb{B}_1 \backslash \mathbb{B}_{1/2}} [|\nabla v|^2 + \delta^{-2} \mathrm{dist}(v, N)^2] dx$ among *functions* $v \in H^1(\mathbb{B}_1 \backslash \mathbb{B}_{1/2}, \mathbb{R}^k)$ with $v = u$ on $\partial(\mathbb{B}_1 \backslash \mathbb{B}_{1/2})$. Since

$$E_\delta(v_\delta) \le \int_{\mathbb{B}_1 \backslash \mathbb{B}_{1/2}} |\nabla u|^2 dx \le \epsilon \le \epsilon_0, \tag{6}$$

we may choose $\delta_i \downarrow 0$ so that $w_i = w_{\delta_i} \to w$ weakly in H^1 and strongly in L^2. The uniform bound on the second term of $E_\delta(v_\delta)$ and the pointwise a.e. convergence of w_i guarantees that $w(x) \in N$ for a.e. x. By the minimality of u, the weak lower semi-continuity of energy, and the minimality of each v_{δ_i},

$$\begin{aligned} \int_{\mathbb{B}} |\nabla u|^2 dx \le \int_{\mathbb{B}} |\nabla w|^2 dx &\le \liminf_{i \to \infty} \int_{\mathbb{B}} |\nabla w_i|^2 dx \\ &\le \limsup_{i \to \infty} \int_{\mathbb{B}} |\nabla w_i|^2 dx \le \int_{\mathbb{B}} |\nabla u|^2 dx, \end{aligned}$$

so that w is also energy-minimizing. Using the penalized energy density

$$e_\delta = |\nabla v_\delta|^2 + \delta^{-2} \mathrm{dist}(v_\delta, N)^2,$$

one may, as before, establish a Bochner inequality

$$\Delta e_\delta \ge -c(n, N) e_\delta (1 + e_\delta)$$

in $\mathbb{B}_1 \backslash \mathbb{B}_{1/2}$ as well as a monotonicity inequality

$$\rho^{2-n} \int_{\mathbb{B}_\rho(a)} e_\delta dx \le \sigma^{2-n} \int_{\mathbb{B}_\sigma(a)} e_\delta dx$$

whenever $\mathbb{B}_\rho(a) \subset \mathbb{B}_\sigma(a) \subset \mathbb{B}_1 \backslash \mathbb{B}_{1/2}$. Combining these with (6) as in the proof of (5) gives the uniform estimate

$$\sup_{\mathbb{B}_{5/6} \backslash \mathbb{B}_{5/8}} e_\delta \le C(n, N)\epsilon$$

provided $\epsilon \le \epsilon_0(n, N)$. Letting $i \to \infty$ shows that

$$\mathrm{osc}_{\partial \mathbb{B}_{3/4}} w \le c \sup_{\partial \mathbb{B}_{3/4}} |\nabla w| \le C \limsup_{i \to \infty} \sup_{\partial \mathbb{B}_{3/4}} |\nabla w_i| \le C\sqrt{\epsilon} \le C\sqrt{\epsilon_0}.$$

Energy minimality readily shows that the w image of the whole ball $\mathbb{B}_{3/4}$ has small diameter and the regularity of $w|\mathbb{B}_{3/4}$ holds as in [69]. Since $u|\mathbb{B}_{1/2} = w|\mathbb{B}_{1/2}$ the conclusion follows from Schoen's estimate (5) applied to w.

Energy minimality is crucial in the above proof as in the original proof of [97] and in the blow-up arguments of [53] and [83]. Such small energy regularity is actually false for a general weakly harmonic map without extra hypothesis. Liao [77] verified it in the special case where 0 is the only singular point. In 1992, L.C. Evans [37] established it for a stationary harmonic map to the standard sphere by exploiting the particular structure of (2), motivated by Hélein's proof [67], and the smallness of the BMO norm. Finally, in 1993, F. Bethuel [7] established small energy regularity for a general stationary harmonic map. He cleverly employs a special frame for the tangent bundle of N along u, as in [68], which gives reformulation of the harmonic map equation using Jacobian expressions that admit compensation properties discovered in [25].

The monotonicity inequality (4) readily implies that the *density* function

$$\Theta_u(a) = \lim_{r\downarrow 0} r^{2-n} \int_{\mathbb{B}_r(a)} |\nabla u|^2 dx$$

exists and is upper semi-continuous in $a \in M$. Clearly Θ_u vanishes at a regular point of a. Moreover, for an arbitrary H^1 function v on $M \subset \mathbb{R}^n$, an elementary covering argument [97] shows that the set

$$Z_v = \{a \in M : \limsup_{r\downarrow 0} r^{2-n} \int_{\mathbb{B}_r(a)} |\nabla v|^2 dx > 0\}$$

necessarily has $n-2$ dimensional Hausdorff measure $\mathcal{H}^{n-2}(Z_v) = 0$.

Thus the small energy regularity theorem of [7] and the higher regularity theory give the partial regularity result of Bethuel

Theorem 3. [7] *Any stationary harmonic map $u : M \to N$, is smooth on $M \backslash Z_u$ where Z_u (as defined above) is closed in M and has $\mathcal{H}^{n-2}(Z_u) = 0$.*

Is this the optimal estimate for stationary maps? In 1984, R. Schoen [96] showed that the singular set of a limit of a convergent sequence of *smooth harmonic maps* has locally finite $\mathcal{H}^{n-2}$ measure. Recently F.H. Lin [81] proved the $n-2$ rectifiability of the singular set Z_u for any stationary harmonic map u. (See some related remarks below in §5, §6.) All known examples of stationary harmonic maps, such as $\frac{x}{|x|} : \mathbb{B}^3 \to \mathbb{S}^2$, have $\dim(Z_u) \leq n-3$.

For an energy-minimizing map u, Schoen and Uhlenbeck prove the stronger estimate $\dim(Z_u) \leq n-3$ by using an induction argument similar to that of Federer [39] and the important notion of a *tangent map* of u at $a \in M$. This is a weak H^1 limit u_0 of a sequence of rescaled maps,

$$u_{r_i}(x) = u(a + r_i x) \quad \text{for} \quad x \in \mathbb{B}_1,$$

corresponding to some sequence $r_i \downarrow 0$. The monotonicity inequality guarantees the existence of a tangent map at each point $a \in M$.

In case u is energy-minimizing, the convergence to a tangent map is shown, in [97], to be strong in H^1. Then, for $0 < s < 1$, the monotonicity identity (4) gives

$$\int_{\mathbb{B}_1\backslash\mathbb{B}_s} |x|^{2-n} \left| \frac{\partial u_0}{\partial r} \right|^2 dx = \lim_{i\to\infty} \int_{\mathbb{B}_1\backslash\mathbb{B}_s} |x|^{2-n} \left| \frac{\partial u_{r_i}}{\partial r} \right|^2 dx$$
$$= \lim_{i\to\infty} r_i^{2-n} \int_{\mathbb{B}_{r_i}(a)} |\nabla u_{r_i}|^2 dx - (sr_i)^{2-n} \int_{\mathbb{B}_{sr_i}(a)} |\nabla u_{r_i}|^2 dx = 0$$

which implies that $\frac{\partial u_0}{\partial r} = 0$ a.e.; that is, u_0 is homogeneous of degree 0. Also $\Theta_{u_0}(0) = \Theta_u(a)$. Moreover, u_0 is a harmonic map and, in fact, energy-minimizing [84].

For a general stationary harmonic map u, the convergence to a tangent map may not be strong in H^1. We will describe below in §6 an example of a stationary harmonic map from $\mathbb{B}^3$ to $\mathbb{S}^2$ which has an isolated singularity at 0, has density $\Theta_u(0) = 16\pi$, but has a constant tangent map at 0.

4. Extension lemmas

The argument of [97] also employs the useful

Lemma 1. ([97], 4.3). *There exist positive constants $\delta = \delta(n, N)$, $c = c(n, N)$, $q = q(n)$ so that for any $\epsilon \in (0, 1)$, $\gamma \in \mathbb{R}^k$, and $g \in H^1(\partial\mathbb{B}^n, N)$ with*

$$\int_{\partial\mathbb{B}} |\nabla_\omega g|^2 d\mathcal{H}^{n-1} \int_{\partial\mathbb{B}} |g - \gamma|^2 d\mathcal{H}^{n-1} \le \delta^2 \epsilon^q \ , \tag{7}$$

there exists a $w \in H^1(\mathbb{B}^n, N)$ with $w|\partial\mathbb{B} = g$ and

$$\int_{\mathbb{B}} |\nabla w|^2 dx \le c\left(\epsilon \int_{\partial\mathbb{B}} |\nabla_\omega g|^2 d\mathcal{H}^{n-1} + \epsilon^{-q} \int_{\partial\mathbb{B}} |g - \gamma|^2 d\mathcal{H}^{n-1}\right).$$

Their proof involves induction on n and a partition of the ball into annular regions of thickness ϵ which are each subdivided into radial cylinders of radius ϵ. One uses homogeneous extension at key points.

Lemma 2. [54] *If N is simply connected, then Lemma 1 is true <u>without the smallness assumption</u> (7).*

Here the proof is somewhat simpler. One may assume $\gamma = \mathcal{H}^{n-1}(\partial\mathbb{B})^{-1} \int_{\partial\mathbb{B}} g d\mathcal{H}^n$ and take $w = \pi \circ v$ where

$$v(x) = \begin{cases} \gamma & \text{for } \ |x| < 1 - \epsilon \\ \epsilon^{-1}(|x| - 1 + \epsilon)g - \epsilon^{-1}(|x| - 1)\gamma & \text{for } \ 1 - \epsilon \le |x| < 1 \end{cases}$$

and π is a projection onto N with center in a suitable $(k-3)$ dimensional Lipschitz complex $K \subset \mathbb{R}^k \backslash N$. The existence of such a K follows from the simple connectivity of N. For example, with $N = \mathbb{S}^2$ one may take $\pi = (\pi_a|\mathbb{S}^2)^{-1} \circ \pi_a$ where $\pi_a(x) = \frac{x-a}{|x-a|}$ for some $a \in \mathbb{B}_{1/2}$. Using this new version, Hardt, Kinderlehrer, and Lin [54] obtain the universal *interior* energy bound,

$$\int_{\mathbb{B}_r} |\nabla u|^2 dx \le \frac{C(n, N)}{1 - r} \quad \text{for } \ 0 \le r < 1$$

for any energy-minimizing map u of $\mathbb{B}_1^n$ into a compact simply connected N. One also finds the compactness result:

Theorem 4. [54] *Any sequence u_i of energy-minimizing maps from M to a simply connected N has a subsequence strongly H^1_{loc} convergent to a map $u \in H^1_{loc}(M, N)$ such that $u|\Omega$ is minimizing for all $\Omega \subset\subset M$.*

Such an interior energy bound may not exist with N not being simply connected. For example, the energy minimizers $(\cos jx_1, \sin jx_1)$ from $\mathbb{B}^n$ to $\mathbb{S}^1$ have unbounded energy on each subdomain.

However, it is true in general that *the weak H^1 limit of energy-minimizers is an energy-minimizer*. S. Luckhaus [83] [84] proved this using his interesting extension

Lemma 3. [83] *There is a $c = c(n, N)$ so that for any pair g, $h \in H^1(\mathbb{S}^{n-1}, N)$ and $\epsilon \in (0, 1)$, there exists $w \in H^1(\mathbb{S}^{n-1} \times [0, \epsilon], N)$ so that $w|\mathbb{S}^{n-1} \times \{0\} = g$, $w|\mathbb{S}^{n-1} \times \{\epsilon\} = h$,*

$$\int_{\mathbb{S}^{n-1}\times[0,\epsilon]} |\nabla w|^2 dx \leq c\epsilon \int_{\mathbb{S}^{n-1}} (|\nabla g|^2 + |\nabla h|^2) d\mathcal{H}^{n-1} + c\epsilon^{-1} \int_{\mathbb{S}^{n-1}} |g-h|^2 d\mathcal{H}^{n-1}$$

and

$$\operatorname{dist}^2(w, N) \leq c\epsilon^{1-n} \left(\int_{\mathbb{S}^{n-1}} |\nabla g|^2 + |\nabla h|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{S}^{n-1}} |g-h|^2 \right)^{\frac{1}{2}} + c\epsilon^{-m} \int_{\mathbb{S}^{n-1}} |g-h|^2.$$

This allows the construction of an annular interpolation so that a competitor for the limiting map may be used to construct a comparison for a minimizer in the sequence. For other handy applications see [103].

5. Some minimizing tangent maps

Singularities in energy-minimizing maps first occur with domain dimension three. In this case, the singularities are isolated in M and any tangent map is the homogeneous extension $\omega\left(\frac{x}{|x|}\right)$ of a smooth harmonic map $\omega : \mathbb{S}^2 \to N$. What are the possible maps ω? If $\dim N = 1$, then N is a circle and ω lifts to a harmonic function which must be constant by the maximum principle. If $\dim N = 2$, then ω is conformal or anti-conformal [70]. Since ω lifts to the oriented cover, we consider the case N is orientable. For genus $(N) \geq 1$, a lifting argument and the maximum principle again show ω to be constant. With genus $N = 0$, the case where N is the standard $\mathbb{S}^2$ is particularly interesting. In 1986, H. Brezis, J.M. Coron, and E. Lieb proved the classification

Theorem 5. [12] *$\omega\left(\frac{x}{|x|}\right)$ is an energy-minimizing map from $\mathbb{B}^3$ to $\mathbb{S}^2$ if and only if ω is an orthogonal rotation of $\mathbb{S}^2$.*

Early work here was motivated by questions about the defect structure of nematic liquid crystals. Note that ω is necessarily the stereographic lift of a rational function of z or $\bar{z}$. Prior to [12], Luskin et al. [24] actually showed numerically, with error estimates, that the homogeneous extension of z^2 is not energy-minimizing. [12] eliminates *any* such higher degree ω by a beautiful splitting-singularity comparison. The degree one ω also must satisfy a balance of mass condition $\int_{\mathbb{S}^2} |\nabla\omega(x)|^2 x d\mathcal{H}^2 x = 0$; otherwise, homogeneous extension from a point moving away from the origin would reduce energy to first order. The only ones left are rotations.

There are now several interesting direct proofs of the minimality of $\frac{x}{|x|}$. In [12] a notion of "minimal connection" (discussed below) is developed. [4] and [27] use the coarea formula. Lin's proof [78], valid for all dimensions $n \geq 3$, is the quickest. It uses the key quantity

$$\kappa(u) = (\operatorname{div} u)^2 - \operatorname{tr}(\nabla u)^2 = \operatorname{div}[(\operatorname{div} u)u - (\nabla u)u],$$

(found in the liquid crystal literature), the pointwise estimate $k(u) \leq (n-2)|\nabla u|^2$, and the identity $\int_{\mathbb{B}} \kappa(u) dx = (n-1)\mathcal{H}^{n-1}(\mathbb{S}^{n-1})$ that is valid for $u \in H^1(\mathbb{B}^m, \mathbb{S}^{n-1})$

with $u|\partial\mathbb{B} = \mathrm{id}$. Integration gives the desired estimate

$$\int_{\mathbb{B}} |\nabla u|^2 dx \geq \frac{n-1}{n-2}\mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = \int_{\mathbb{B}} \left| \nabla \left(\frac{x}{|x|}\right) \right|^2 dx.$$

Theorems 4 and 5 lead to interior bounds on the number of singularities [5, 58]. F.J. Almgren and E. Lieb [5] also obtain bounds in terms of the energy of the boundary data.

Classifying the minimizing tangent maps in higher dimensions is difficult. Gulliver and Coron verified that the Hopf map $H : \mathbb{S}^3 \to \mathbb{S}^2$ gives the energy-minimizing map $H\left(\frac{x}{|x|}\right) : \mathbb{B}^4 \to \mathbb{S}^2$. Here one also has all the *line-singularity* examples $\omega((x_1^2 + x_2^2 + x_3^2)^{-1/2}(x_1, x_2, x_3))$ corresponding to rotations ω of $\mathbb{S}^2$. Verifying that all tangent maps reduce to constants in special cases leads to improved regularity and partial regularity results [97, 99]. For example, their work implies the

Theorem 6. *If there is no nonconstant harmonic map from $\mathbb{S}^j$ to N for $j = 2, 3, \ldots, n = \dim M$, then any energy-minimizing map $u : M \to N$ is smooth and satisfies*

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C(\Omega, N)$$

whenever $\Omega \subset\subset M$.

Under the same hypotheses on the target N, F.H. Lin [81] recently derived this estimate for a general *stationary* harmonic map. His argument involves Bethuel's small energy regularity theorem [7] and consideration of scaling and limit properties of the measure $|\nabla u|^2 dx$. See the discussion after Theorem 7 below.

Energy-minimizing maps enjoy *complete boundary regularity*, [98, 57, 72] with e.g. Ω and g being $C^{1,\alpha}$ smooth, partly because the boundary tangent maps are necessarily constant. M. Fuchs [41, 42] proved partial regularity results for a *generalized obstacle problem* where ∂N may be nonempty. Also R. Gulliver and J. Jost [49], F. Duzaar and K. Steffen [32, 33], and Hardt and Lin [59] consider a *partially free boundary condition* in which the Dirichlet condition for g on a subset A of ∂M is relaxed to a constrained Neumann condition that $g(A)$ lie in a submanifold Σ of N. Some other interesting constrained problems are [14, 30, 31, 52]. With the target N being complete but possibly noncompact, M. Li [76] proved the (optimal) $n-2$ dimensional estimate on the singular set of an energy minimizer.

Examples of *nonuniqueness* are not difficult to produce. There is a whole 1 parameter family of distinct minimizing maps from $\mathbb{B}^3$ to $\mathbb{S}^2$, each with boundary values $g : \mathbb{S}^2 \to \mathbb{S}^2$ being the stereographic lift of z^2 [55]. However, L. Mou [86] verified uniqueness of minimizers for "almost all" smooth boundary data.

6. Continuous maps and relaxed energy

Singularities in harmonic map problems may be caused by topological obstructions. This occurs with energy minimizers whenever the Dirichlet boundary data admits a finite energy extension but no continuous extension, e.g. $g = \mathrm{id}_{\mathbb{S}^2}$. What if there is no topological obstruction? In [56] is an example of a smooth degree zero map $g : \mathbb{S}^2 \to \mathbb{S}^2$ such that

$$\begin{aligned}(8) \qquad & \inf\left\{\int_{\mathbb{B}} |\nabla u|^2 dx : u \in H^1(\mathbb{B}, \mathbb{S}^2),\ u|\partial\mathbb{B} = g\right\} \\ & < \inf\left\{\int_{\mathbb{B}} |\nabla v|^2 dx : v \in C^0(\mathbb{B}, \mathbb{S}^2),\ v|\partial\mathbb{B} = g\right\}.\end{aligned}$$

Any energy-minimizer achieving the left-hand side must have singularities even though there exists a smooth extension of g. Here for a point a near the North Pole in $\mathbb{B}^3$, g is approximately $\frac{x-a}{|x-a|}$ near the North Pole, $-\frac{x+a}{|x+a|}$ near the South Pole, and identically $(0,0,-1)$ in the large meridian region. The gap between the infima in (8) is approximately $8\pi(2|a|)$.

Is the infimum on the right-hand side achieved? By making the example of [56] axially symmetric and using a result of D. Zhang [121], one can get a smooth harmonic extension. However, it is still an open question: *Does every smooth degree* 0 *map from* $\mathbb{S}^2$ *to* $\mathbb{S}^2$ *extend to a smooth harmonic map from* $\mathbb{B}^3$ *to* $\mathbb{S}^2$? There is also the more vague question: *What happens when one tries to minimize in a class of continuous maps*?

The work [8] of F. Bethuel, H. Brezis, and J.M. Coron is fundamental to this question. They prove the infimum on the right-hand side of (7) is, for a general degree zero g, equal to

$$\inf\{F(u) : u \in H^1(\mathbb{B}, \mathbb{S}^2), u|\partial\mathbb{B} = g\}$$

where

$$\begin{aligned} F(u) &= \int_{\mathbb{B}} |\nabla u|^2 dx + 8\pi L(u), \\ L(u) &= \frac{1}{4\pi} \sup_{\xi:\mathbb{B}\to\mathbb{R}, |\nabla\xi|\le 1} \left\{\int_{\mathbb{B}} [D(u)\cdot\nabla\xi] dx - \int_{\partial\mathbb{B}} [D(u)\cdot x]\xi dx\right\}, \\ D(u) &= \left(u\cdot\frac{\partial u}{\partial x_2}\wedge\frac{\partial u}{\partial x_3},\ u\cdot\frac{\partial u}{\partial x_3}\wedge\frac{\partial u}{\partial x_1},\ u\cdot\frac{\partial u}{\partial x_1}\wedge\frac{\partial u}{\partial x_2}\right). \end{aligned}$$

Here, for u in the (H^1 strongly dense) class of maps from $\mathbb{B}^3$ to $\mathbb{S}^2$ having only finitely many singularities, each of degree ± 1, $L(u)$ is simply the length of the "minimal connection" or 1-chain whose boundary is the 0-chain of singular points oriented by their degrees. They prove

Theorem 7. [6, 8] (1) *L is H^1 strongly continuous,* (2) $L(u) = 0$ *if and only if u is a strong H^1 limit of continuous maps, and* (3) $F(u)$ *is weakly H^1 lower semi-continuous.*

Thus F may be minimized and [8] in fact finds infinitely many distinct harmonic extensions for any boundary data g exhibiting the gap phenomenum of [56]. Since L in some sense penalizes singularity, one could hope to find smooth harmonic maps by minimizing F.

What is the (partial) regularity of an F minimizer? M. Giaquinta, G. Modica, and Soucêk proved in [44] that sing u is a closed rectifiable set of finite $\mathcal{H}^1$ measure. Their proof involved their important general program [45] of deriving and understanding the relaxed energy F using a completion of the graph of u called a Cartesian current. The Federer-Fleming compactness theorem for integral currents [38] and a monotonicity formula for F provide the crucial small F regularity lemma.

Is the singularity estimate of [44] *optimal*? In the restricted context of axially symmetric maps for $\mathbb{B}^3$ to $\mathbb{S}^2$, Hardt, Lin, and Poon [64] showed that the singular set of an F minimizer is at most an isolated set (in the axis). On the other hand, by using a second gap phenomenum, [64] shows that a singularity actually can occur for an F minimizer in the axially symmetric class. Here the singularity has degree 0 and the tangent map is constant. By imposing a reflection symmetry, one obtains a balance of energy condition which is sufficient to insure that the map is also a stationary harmonic map. This example suggests that F minimization is not sufficient to produce smooth harmonic extensions. Ironically this work has led to more success guaranteeing singularity rather than regularity in examples.

In [89], C. Poon constructed, for any point $a \in \overline{\mathbb{B}}^3$, a harmonic extension $v_a \in H^1(\mathbb{B}, \mathbb{S}^2) \cap C^\infty(\overline{\mathbb{B}} \backslash \{a\})$. By minimizing $E + 8\pi L(u - v)$ for a suitable given v as in [8], [64] shows how one may *prescribe* not only the Dirichlet data g, but also a finite set of singularities on the axis. T. Rivière successfully iterated this construction to produce in [92] a line of singularities. In [93] he even constructed an *everywhere singular* weakly harmonic map. His key lemma is related to the [56] example. There one verified that the energy required to cancel 2 singularities $a_\pm$ of degrees ± 1 was $\Lambda = 8\pi \mathrm{dist}(a_+, a_-)$. Similarly one can *insert* two such nearby singularities in a given smooth map w by increasing the energy by approximately Λ. Rivière proves that one can, for nonconstant w, in fact do this with an energy increase *strictly less than* Λ. This key lemma was suggested in an early version of [8] and verified first in the axially-symmetric case in [64].

In general dimensions, the behavior of minimizing sequences of continuous maps is treated in the striking recent work [82] of F.H. Lin.

Theorem 8. [82] *Suppose that* $g : \partial M \to N$ *is smooth and*

$$\mathcal{U} = \{u \in H^1(M, N) \cap C^0 : u|\partial M = g\}$$

is nonempty. Then any energy-minimizing sequence in $\mathcal{U}$ *has a subsequence* H^1 *weakly convergent to a harmonic map* u *which is smooth away from a closed* $n-2$ *rectifiable subset* Z *of* M *with* $\mathcal{H}^{n-2}(Z) < \infty$*. If, in addition,* $\pi_2(N) = 0$*, then* u *is absolutely energy-minimizing* (*so that* $\dim Z \le n-3$)*.*

Here rectifiability means that Z is, except for an $\mathcal{H}^{n-2}$ measure zero set, contained in a countable union of $n-2$ dimensional C^1 submanifolds [38]. In proving this theorem, Lin examines the behavior of the measures $\mu_i = |\nabla u_i|^2 dx$ corresponding to the minimizing sequence u_i. Passing to a subsequence $u_i \to u$ weakly in H^1 and $\mu_i \to \mu = |\nabla u|^2 dx + \nu$ as measures. The *defect measure* ν is nonnegative and Radon. Lin proves a small energy regularity theorem for μ. That is, *if* $r^{2-n}\mu(\mathbb{B}_r(a)) \le \epsilon_0$, then on $\mathbb{B}_{r/2}(a)$ u *is smooth and* ν *vanishes*. His proof uses a blow-up argument as in [53], [83], [41] and the following lemma (whose analogue in the energy-minimizing case follows from Lemma 1).

Lemma 4. *If* $r^{2-n}\mu\mathbb{B}_r(a) \le \epsilon_0$ *and* $0 < \lambda < 1/2$*, then*

$$\left(\frac{r}{2}\right)^{2-n} \mu\mathbb{B}_{r/2}(a) \le \lambda r^{2-n}\mu\mathbb{B}_r(a) + c(\lambda, m, N) r^{-n} \int_{\mathbb{B}_r(a)} |u - \bar{u}_r|^2 dx.$$

One may essentially repeat the ([97], 4.3) proof, using the minimizing sequence. While homogeneous extension does destroy continuity, one may instead, for example, replace a map $u \in H^1(\mathbb{B}, N) \cap C^0(\overline{\mathbb{B}})$ by the continuous map

$$w_\delta = \begin{cases} u(x/|x|) & \text{for } \ x \in \mathbb{B}_1 \backslash \mathbb{B}_\delta \\ u(x/\delta) & \text{for } \ x \in \mathbb{B}_\delta \end{cases}$$

so that $E(w_\delta) = E\left(u\left(\frac{x}{|x|}\right)\right) + \delta^{n-2}E(u)$ and then choose δ small. Lin also verifies a monotonicity inequality for $r^{2-n}\mu\mathbb{B}_r(a)$. This along with the small energy regularity shows that the density $\lim_{r\downarrow 0} r^{2-n}\mu\mathbb{B}_r(x)$ exists and is $\geq \epsilon_0$ precisely on $Z = M \cap \operatorname{sing} u$. A result of D. Preiss [90] (with a simple proof in [82] of this application) gives the rectifiability. If Z is nonempty, then an induction argument and a rescaling argument as in [95] produces a nonconstant harmonic map from $\mathbb{S}^2$ to N. This is only possible if $\pi_2(N) \neq 0$.

7. Uniqueness of the tangent map

An important question is whether a tangent map $u_0 = \lim_{i\to\infty} u(a + r_i(\cdot))$ is *unique*, that is, depending on the original map u and the point a, but independent of the rescaling sequence $r_i \downarrow 0$. In case $\operatorname{sing} u_0 = \{0\}$ so that a is an *isolated* singularity for u, such uniqueness would imply that the difference

$$u(x) - u_0(x - a)$$

is continuous and vanishes at a. For this case, L. Simon proved the following fundamental result in 1983:

Theorem 9. [101] *Suppose $u : M \to N$ is an energy-minimizing map, N is real analytic, and $a \in M \cap \operatorname{sing} u$. If u_0 is a tangent map of u at a with $\operatorname{sing} u_0 = \{0\}$, then u_0 is the unique tangent map at a.*

The original proof in [101] of this theorem was technically rich and interesting. Here we describe a recent shorter proof of L. Simon [107]. A key feature is again the Lojaciewicz inequality. This gives, for any real analytic function f and critical point y of f, positive numbers σ and $\alpha < 1$ so that

$$|\nabla f(x)| \ \geq \ |f(x) - f(y)|^{1-\frac{\alpha}{2}} \qquad \text{whenever} \ \ |x - y| < \sigma.$$

From this Simon derives a useful infinite-dimensional analogue. Note that $\varphi = u_0|\mathbb{S}^{m-1}$ is a critical point for the energy functional

$$E(\psi) = \int_{\mathbb{S}^{n-1}} |\nabla_\omega \psi|^2 d\mathcal{H}^{n-1} \qquad \text{for} \ \ \psi \in H^1(\mathbb{S}^{n-1}, N),$$

and that $\mathcal{M} = \nabla E$ is the harmonic map (Euler-Lagrange) operator; hence

$$\mathcal{M}\psi = \Delta_{\mathbb{S}^{n-1}}\psi - A_N(\psi)(\nabla_\omega\psi, \nabla_\omega\psi).$$

Then Simon's Lojaciewicz-inequality is

$$\|\mathcal{M}\psi\|_{L^2(\mathbb{S}^{n-1})} \ \geq |E(\psi) - E(\varphi)|^{1-\frac{\alpha}{2}} \qquad \text{for} \ \ \|\psi - \varphi\|_{L^2(\mathbb{S}^{n-1})} < \sigma.$$

The rescaled maps u_{ρ_i}, being pointwise bounded and converging weakly in H^1, converge strongly in L^2. Thus, for any positive η (determined later), we have

$$\int_{\mathbb{B}_{3/2}\backslash\mathbb{B}_{3/4}} |u_\rho - u_0|^2 dx < \eta^2 \tag{9}$$

for $\rho = \rho_i$ and i sufficiently large. By an initial rescaling we assume for convenience that (9) holds with $\rho = 1$. We next assume $\tilde{u} = u_\rho$ is some fixed rescaling, again satisfying (9). For η sufficiently small, one may apply (9) and an estimate from [97] (which readily follows from Lemma 1 and Theorem 2) on any ball of radius $\frac{3}{8}$ in the annulus $\mathbb{B}_{3/2}\backslash\mathbb{B}_{3/4}$. This and standard interior estimates give that

$$\|\tilde{u} - \psi\|_{C^3(\mathbb{B}_{5/4}\backslash\mathbb{B}_{7/8})} < \gamma = \gamma(n, N, \eta). \tag{10}$$

The monotonicity identity, comparison with the map $w = \tilde{u}\left(\frac{x}{|x|}\right)$, and the Lojaciewicz inequality imply

$$\begin{aligned} \int_{\mathbb{B}_1} r^{2-n} \left| \frac{\partial \tilde{u}}{\partial r} \right|^2 dx &= \int_{\mathbb{B}_1} |\nabla \tilde{u}|^2 dx - \Theta_{\tilde{u}}(0) \\ &\le \int_{\mathbb{B}} |\nabla w|^2 dx - \Theta_{u_0}(0) \\ &= \frac{1}{n-2}[E(\psi) - E(\varphi)] \\ &\le \frac{1}{n-2}\|M(\psi)\|_{L^2(\mathbb{S}^{n-1})}^{\frac{2}{2-\alpha}} \end{aligned} \tag{11}$$

where $\psi = \tilde{u}|\mathbb{S}^{n-1}$. Writing out the harmonic map equation for $\tilde{u}$ in spherical coordinates $r = |x|$, $\omega = \frac{x}{|x|}$ gives

$$r^{1-n}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial \tilde{u}}{\partial r}\right) + r^{-2}\Delta_{\mathbb{S}^{n-1}}\tilde{u} + r^{-2}A_N(\tilde{u})\left(\nabla_\omega \tilde{u}, \nabla_\omega \tilde{u}\right) + A_N(\tilde{u})\left(\frac{\partial \tilde{u}}{\partial r}, \frac{\partial \tilde{u}}{\partial r}\right) = 0.$$

Restricting to $\mathbb{S}^{n-1}$, the two middle terms become $\mathcal{M}(\psi)$. So combining with (7) and (11) gives

$$\begin{aligned} \left(\int_{\mathbb{B}_1} r^{2-n} \left| \frac{\partial \tilde{u}}{\partial r} \right|^2 dx\right)^{2-\alpha} &\le C\|\mathcal{M}(\psi)\|_{L^2}^2 \\ \le C \int_{\mathbb{S}^{n-1}} &\left(\left| \frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial \tilde{u}}{\partial r}\right) \right|^2 + \left| \frac{\partial \tilde{u}}{\partial r} \right|^2 \right) d\mathcal{H}^{n-1}. \end{aligned} \tag{12}$$

To handle the second derivative term, we observe that the domain deformation $\tilde{u}((1+t)x)$ is again a harmonic map. Thus $\frac{d}{dt}\Big|_{t=0} \tilde{u}((1+t)x) = r\frac{\partial \tilde{u}}{\partial r}$ satisfies the associated (linear) Jacobi field equation. Using (11) and standard interior elliptic estimates shows that

$$\sup_{\mathbb{S}^{m-1}} \left| \frac{\partial}{\partial r}\left(r\frac{\partial \tilde{u}}{\partial r}\right) \right|^2 \le C \int_{\mathbb{B}_{5/4}\backslash\mathbb{B}_{7/8}} \left| \frac{\partial \tilde{u}}{\partial r} \right|^2 dx.$$

Combining with (12) gives the crucial estimate

$$\left(\int_{\mathbb{B}_1} r^{2-n} \left| \frac{\partial \tilde{u}}{\partial r} \right|^2 dx\right)^{2-\alpha} \le C \int_{\mathbb{B}_{3/2}\backslash\mathbb{B}_{3/4}} r^{2-n} \left| \frac{\partial \tilde{u}}{\partial r} \right|^2 dx.$$

If (9) is actually valid for *all* scales $\rho \in [\sigma, 1]$, then some elementary algebra and an iteration argument lead to an estimate

$$\int_{\mathbb{B}_\sigma} r^{2-n} \left| \frac{\partial \tilde{u}}{\partial r} \right|^2 dx \le \frac{C}{|\log \sigma|^{1+\beta}}$$

where $\beta = (1-\alpha)^{-1} > 0$. Some calculus and Cauchy-Schwartz then imply

$$\|u_\sigma - u_\tau\|_{L^2(\mathbb{S}^{n-1})} \le \int_\sigma^\tau \left\|\frac{\partial u_r}{\partial r}\right\|_{L^2(\mathbb{S}^{n-1})} dr$$

$$\le \left(\int_\sigma^\tau r|\log r|^{1+\frac{\beta}{2}}\left\|\frac{\partial u_r}{\partial r}\right\|^2 dr\right)^{\frac{1}{2}}\left(\int_\sigma^\tau r^{-1}|\log r|^{-1-\frac{\beta}{2}}dr\right)^{\frac{1}{2}} \le C|\log\tau|^{-\frac{\beta}{2}}$$

for each $\tau \in [\sigma, 1]$. Suitable choices of the constants now allow us to verify (9) for all $\rho \in (0,1)$ and to obtain the desired decay

$$\|u_\rho - \varphi\|_{L^2(\mathbb{S}^{m-1})} \le C\|\log\rho\|^{-\frac{\beta}{2}},$$

which implies uniqueness of the tangent map $u_0 = \varphi\left(\frac{x}{|x|}\right)$.

The logarithmic decay here may be improved to a positive power decay provided the Jacobi fields along the given harmonic map close to φ are uniformly integrable, i.e., representable as the initial velocity of a one-parameter family of harmonic maps. See the nice treatment in [102]. Such a criterion first arose in Almgren and Allard's study of tangent cones to minimal surfaces [2]. With dimensions $n = 3$ and $\dim N = 2$ where harmonic maps from $\mathbb{S}^2$ to N are conformal or anti-conformal, R. Gulliver and B. White [50] have verified this integrability criterion. But it seems difficult to check in higher dimensions, and it fails in related problems [1].

What about the analyticity assumption in [101]? It is crucial. In [119] B. White gave an example of *nonuniqueness* of a tangent map at an isolated singularity of a harmonic map into a smooth (nonanalytic) N. His construction actually produces a whole 1 parameter family of distinct tangent maps.

In dimensions ≥ 4, higher-dimensional singularities may occur in energy-minimizing maps, and their structure and asymptotics present exciting challenges. L. Simon provided in [104, 105, 109] results concerning the uniqueness of the tangent map or of the tangent cone for minimal surfaces where the tangent object includes a whole line of singularities. Here, the tangent cone may be the product of a line (or higher-dimensional space) with a minimizing cone X having an isolated singularity. Analogously the tangent map may be independent of some variables and have an isolated singularity with respect to the others. Assuming a Jacobi-field integrability condition and some other conditions (including strict stability, strict minimality) for X, he proved in [105] uniqueness of a product cone $X \times \mathbb{R}^\ell$. Similar uniqueness results for harmonic maps occur in [109]. In one special case, much can be said about the topology of the singular set.

Theorem 10. [60] *If $u : \mathbb{B}^4 \to \mathbb{S}^2$ is an energy-minimizing map with smooth boundary data, then $\operatorname{sing} u$ consists of a finite set and finitely many Hölder continuous closed curves with only finitely many crossings.*

The local $C^{1,\alpha}$ regularity of these curves requires the uniqueness results of [109]. It is unknown if the singular set here has finite $\mathcal{H}^1$ measure, as there may be some oscillation near the crossings. The proof of the above theorem uses a description of possible tangent maps and the Reifenberg topological disk lemma [91]. The latter states roughly that any compact subset of $\mathbb{R}^m$ that is at *all* small scales close, in the Hausdorff metric, to some k-plane is locally a Hölder continuous submanifold. The fact that $\operatorname{sing} u$ is usually locally in a small neighborhood of some line follows from the small energy regularity lemma. The opposite fact that the approximating line is in a small neighborhood of $\operatorname{sing} u$ is caused by a topological obstruction and is special to the $(\mathbb{B}^4, \mathbb{S}^2)$ problem.

For more general targets, the singular set may conceivably have many small gaps. Nevertheless, L. Simon [106] recently proved the remarkable general theorem.

Theorem 11. [106] *If $u : M \to N$ is energy-minimizing with N compact and real analytic, then for each closed ball $B \subset M$, $B \cap \operatorname{sing} u$ is the union of a finite pairwise disjoint collection of locally $(n-3)$-rectifiable locally compact sets.*

There are several new ideas in Simon's proof of this theorem which we describe here only vaguely. (See also [107, 108].) Let $m = n-3$. The first is a new technical criteria for the m-rectifiability of a set involving an alternative at each scale between being located near some m-plane and having fixed size gaps. Proving this criteria involved a new covering lemma for a subset F of $\mathbb{R}^m$ by balls containing a fixed size subball missing F.

Following refinement of an argument of Almgren [3], one reduces to considering singular points a at which occurs a tangent map φ depending only on three coordinates and for which $\operatorname{sing} \varphi$ is an $m = n-3$ dimensional plane. The analyticity is used to show that there are only finitely many possibilities $\{\alpha_1, \dots \alpha_N\}$ for $\Theta_u(a) = \Theta_\varphi(0)$. It then suffices to show each set

$$\mathbb{S}_j^+ = \{z \in \operatorname{sing} u : \Theta_u(z) \geq \alpha_j\}$$

(which is closed by the upper-semicontinuity of Θ_u) is locally m rectifiable.

In the proof of this, bookkeeping concerning the holes in $\mathbb{S}_j^+$ is facilitated by a measure μ which is defined roughly as a sum of $\mathcal{H}^m$ restricted to the subset with no significant gaps and a weighted sum of point masses associated with various size gaps. The resulting measure enjoys upper and lower m-dimensional density bounds.

Recall the key quantity $r^{2-n}|\frac{\partial u}{\partial r}|^2 = |x|^{-n}|x \cdot \nabla u|^2$ was used in the above isolated singularity discussion. Now Simon averages this, using the measure μ to define the deviation function

$$\psi(x) = \int_{\mathbb{S}_j^+} |x-a|^{-n}|(x-a) \cdot \nabla u(x)|^2 d\mu(a).$$

The critical estimate established here is

$$\int_{T_{\theta_\rho}} \psi(x)dx \ \leq \ C \left(\int_{T_\rho \setminus T_{\theta_\rho}} \psi(x)dx \right)^{1/2-\alpha}.$$

Here, for $0 < \rho < \frac{1}{4}$ and some fixed $\delta > 0$, T_ρ is a union of radius ρ balls centered in $\mathbb{S}_j^+$ in which there is an approximating plane of small tilt contained in a $\delta\rho$ neighborhood of $\mathbb{S}_j^+$. The proof of this estimate involves a variety of energy estimates, some using a Lojaciewicz type inequality, as well as L^2 estimates concerning u and its tangent map. The paper [106] is certainly technically hard, but the richness of the ideas and the strength and applicability of the results will reward the conscientious reader.

8. Some related areas

There are some important and active related areas, about which our comments will have to be brief and our references few.

Replacing $|\nabla u|^2$ by $|\nabla u|^p$, where $1 < p < \infty$, gives rise to *p-energy* and *p-harmonic maps*. The partial regularity theory of minimizers was treated by S. Luckhaus [83], Hardt and Lin [57], and M. Fuchs [41], [42] and of stationary maps to the sphere by [87]. These use the $C^{1,\alpha}$ regularity theory for p-harmonic equations and systems (e.g. [113], [112]) for which there is a large literature. Some restricted classes of maps are treated in [29, 116, 118], and some applications to geometry are given by W. Wei [117].

The map $\frac{x}{|x|}$ is p-energy minimizing for $p \in \{2, 3, \ldots, n-1\}$ [27], for $p \in [n-1, n)$ [66], and for $p \in [2, n - 2\sqrt{n-1}]$ [115]. *Is it true for the remaining p's*? For $p \in [n-1, n)$ R. Musina earlier [88] had verified the p-energy minimality of $\mathrm{id}_{\mathbb{S}^{n-1}}$ among degree 1 maps. Also for $p \in (n-1, n)$ one may minimize p-energy while prescribing singularities with degrees [17]. As $p \uparrow n$, the gap phenomenon for singularities of minimizers disappears [62], [65]. Singularities reduce to the topologically minimum number $|\deg g|$, and the p-energy approaches ∞. However, subtracting off the term $|\deg g|\mathcal{H}^{n-1}(\mathbb{S}^{n-1})\frac{(n-1)^{p/2}}{n-p}$ gives, as $p \uparrow n$, a renormalized energy determined only by g and the location of the singularities. Passing to a subsequence, the p_i-energy minimizers converge, away from the limit location of their singularities.

Such phenomena was first discovered with the Ginzburg-Landau functional in 2 dimensions by F. Bethuel, H. Brezis, and F. Hélein [9], [10], [111]. There is now a vast growing literature on Ginzburg-Landau. See e.g. the book [10] and recent survey [80]. Higher dimensional singularities occur in a Ginzburg-Landau problem in [94, 82] and in the n-harmonic problem in [63].

The study of harmonic maps to various singular spaces is also a growing area of research. Motivated by important applications to algebraic geometry, most work has focused on the existence of regular (e.g. Lipschitz) solutions which usually map to nonpositively curved metric spaces [46, 71, 73, 75, 100, 120]. A model for nematic liquid crystals leads to harmonic maps to a metric cone over $\mathbb{S}^2$ or $\mathbb{R}P^2$ [79, 61] which may exhibit curve singularities. Partial regularity for minimizing maps to (possible infinitely curved) polyhedra is found in [114]. Harmonic maps with singular domains are studied in [18, 73, 81].

The harmonic map heat flow equation is

$$\frac{\partial u}{\partial t} - \Delta_M u = A_M(u)(\nabla u, \nabla u).$$

For a compact negatively curved smooth target N, the existence of a regular solution of the initial value problem was obtained by Eells and Sampson for $\partial M = \emptyset$ and by R. Hamilton [51] for $\partial M \neq \emptyset$ with Dirichlet and Neumann data. In 1985 M. Struwe [110] studied the general problem in two space dimensions. His solution was regular except for a possible finite set of space-times. The first example of singularity in two dimensions was given by K.C. Chang, Y. Ding, and R. Ye [15] in 1992, although there were other higher-dimensional examples of finite-time blow up with smooth initial data [16, 47]. The existence, in higher dimensions, of a partially regular weak solution was given by Y. Chen and M. Struwe [22] in 1989. See Cheng [23] for a slight improvement and Chen-Lin [20] for the Dirichlet problem. In case N is a sphere, a time-step iteration scheme suggested by Y. Kikuchi [74, 11] works. Following [26] or [11], there is no uniqueness for the initial value problem. While Rivière's example [93] indicates there is no regularity for a general harmonic map heat flow, there may be partial regularity with an extra condition. This is true for example if $N = \mathbb{S}^{k-1}$ and the solution is energy nonincreasing in time and satisfies the Struwe's parabolic monotonicity inequality [40, 21]. As with harmonic maps, one has the important (but imprecise) question: *What is the nature of and asymptotics near singularities for various restricted classes of harmonic map flows*? For 2-space dimensions, some answers are given by Struwe [110].

Practically all the issues we have discussed recur in other geometric and applied variational problems. There has already been and will continue to be many very rich interchanges. Because of the simplicity and beauty of the singularity theory to date as well as the wide horizon of possibility in varying the geometry of both domain and range, singular harmonic maps will certainly, like regular harmonic maps before, continue to flourish.

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