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The classification of the finite simple groups, by Daniel Gorenstein, Richard Lyons, and Ronald Solomon, Mathematical Surveys and Monographs, vol. 40, no. 2, American Mathematical Society, Providence, RI, 1994, xiv + 218 pp., $59.00, ISBN 0-8218-0390-5

This is the second volume in a multipart series which seeks to provide a complete, self contained, and simplified proof of the theorem classifying the finite simple groups. The reader is directed to Ron Solomon's expository article on the Classification appearing in the February 1995 issue of the *Notices* [S] for a description of the Gorenstein-Lyons-Solomon (GLS) program. My review of volume one in the series (cf. the October 1995 issue of the *Bulletin*) sought to place the GLS program in a context which would help explain its importance. Volume one [GLS1] gives an outline of the GLS proof. Volume two is entitled *General group theory*. The title is chosen to emphasize the fact that the theorems in volume two apply to all finite groups, rather than to some specific simple group on the list $\mathcal{K}$ of known simple groups.

This dichotomy between general group theory and the study of the groups in $\mathcal{K}$ using special properties of such groups is important and requires some explanation. Since the Classification, a typical problem in finite group theory is usually solved as follows: First, reduce the problem to a problem about suitable minimal groups. (Often simple groups.) Second, use special facts about the minimal groups to complete the solution. Local group theory is usually the major tool in the reduction process. Facts about the structure and representations of the simple groups are crucial in analyzing minimal groups.

"So what?" you might ask. "Isn't GLS devoted to a new proof of the Classification, not to the solution of post Classification problems?" The answer is "Yes, but..." The general group theory in volume two of GLS is chosen specifically to provide a foundation for the GLS proof of the Classification, but a significant fraction is also relevant in reducing problems about general finite groups to problems about simple groups. This should not be surprising. The Classification is valuable precisely because it allows us to make effective use of the hypothesis of simplicity via the study of properties of an explicit list $\mathcal{K}$ of simple groups. The proof of the Classification is long and complicated because it is difficult to make use of simplicity by any means other than an appeal to properties of groups in $\mathcal{K}$. The techniques developed to cope with these difficulties are also the most effective techniques for analyzing the general finite group in terms of its composition factors.

As is often the case in mathematics, it is possible to prove a very strong theorem like the Classification even though it is impossible to prove weaker theorems directly. This is because the proof of the Classification is inductive, so that one may assume all composition factors of proper subgroups of a minimal counterexample are in $\mathcal{K}$ and hence use properties of the groups in $\mathcal{K}$. Weaker theorems usually do not lend themselves to such strong inductive constraints on proper subgroups. A significant fraction of the proof of the Classification is devoted to establishing properties of the groups in $\mathcal{K}$ which translate into statements about the proper subgroups of a

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minimal counterexample to the Classification, using the general group theory in GLS2.

Local group theory is the most important tool for studying the general finite group. The volume under review is primarily devoted to a discussion of local group theory. I will therefore digress for a brief discussion of local group theory and its history.

Local group theory studies a finite group G from the point of view of so-called *local subgroups* of G: that is, the normalizers of nontrivial p-subgroups of G. Sylow's Theorem is the first important local group theoretic result. After Sylow there was a long hiatus during which the subject was virtually dormant. (Ron Solomon disagrees slightly, pointing to work of Burnside at the turn of the century.) Then in the fifties, local group theory began to return to life with the Hall-Higman Theorem, Suzuki's early papers on simple groups, and Brauer's proposal that finite simple groups should be characterized in terms of centralizers of involutions. At the end of the fifties and the beginning of the sixties came Thompson's thesis proving the nilpotence of Frobenius kernels and the Odd Order Paper of Feit and Thompson, which introduced new methods that revolutionized finite group theory and began a 20-year period of intense activity, culminating in the classification of the finite simple groups in about 1980.

Local group theory plays by far the most important role in the proof of the Classification. From 1960 to 1980 it was the major focus of interest in finite group theory. After the Classification, activity in the field dropped sharply, although local group theory remains important because of its role in reduction theory.

Probably the *generalized Fitting subgroup* $F^*(G)$ is the most important tool for analyzing the general finite group G. In essence $F^*(G)$ is the smallest normal subgroup of G that controls the structure of G. On the other hand, the structure of $F^*(G)$ is relatively uncomplicated, and it is possible to relate $F^*(G)$ to $F^*(H)$ for suitable subgroups H of G.

The notion of the generalized Fitting subgroup has it roots in work of Fitting on solvable groups and Wielandt on subnormal subgroups. In the late sixties, Gorenstein and Walter contributed the idea of a "component" of a finite group. The actual definition of the generalized Fitting subgroup came in the early seventies and is due to Helmut Bender, who also gave an elegant treatment of the notion.

Somewhere between one quarter and one third of GLS2 is devoted to the generalized Fitting subgroup and related topics. There is also a discussion of basic techniques like 1-cohomology, transfer, and fusion, plus sections on more sophisticated tools like signalizer functors and Thompson factorization. There are sections discussing geometric techniques that have evolved since the Classification which study groups from the point of view of presentations as amalgamated products.

The authors are among the best expositors in finite group theory. Their treatment of all these topics is clear and elegant. My only quarrel with the presentation is the decision by the authors not to motivate most of the results they treat. Without a good picture of their approach to the Classification, it is difficult to understand the significance of many of the lemmas and theorems that appear. Volume one of GLS supplies an outline of the proof, but the outline is long and complicated and unlikely to be fresh in the reader's mind by the time he or she gets to volume two, even if the reader chooses to start with volume one. It would also be useful to point out results that are of interest independent of the Classification.

Volume two of the GLS series gives an attractive treatment of local group theory. Readers who are not experts on finite group theory would probably be well served by first reading one of the texts listed in the basic references in GLS2 for an initial exposure to the fundamentals of local group theory. In most instances GLS2 records basic results from those texts without proof but with a reference and then goes much deeper into the subject while still discussing important material of fairly wide interest. The authors have chosen topics specifically to support their proof of the Classification, but much of GLS2 is useful in reducing problems about general finite groups to problems about simple groups. For this reason, GLS2 should be in the library of all finite group theorists, even those who do not wish to read the GLS proof of the Classification.

References

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[S] R. Solomon, *On finite simple groups and their classification*, Notices AMS **42** (1995), 231– 239. MR **96c:**20001

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