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Linear operators and ill-posed problems, by M. M. Lavrentév and L. Ya. Savelév, translated from Russian by Nanka Publishers, Moscow, Consultants Bureau, Plenum Publishing Corporation, New York, 1995, xii + 382 pp., $110.00, ISBN 0-306-11035-0

On page 1 of M. M. Lavrentév's Springer Tract #11 [12], one finds a reasonably broad definition of a well-posed problem. Namely, let Φ, F be some complete metric spaces, and let $A\varphi$ be a function with domain of definition Φ and range F. Consider the equation

$$A\varphi = f. \tag{1}$$

We say that the problem of solving (1) is properly (well) posed if

(i) the solution φ of (1) exists for any $f \in F$,
(ii) the solution of φ of (1) is unique in Φ,
(iii) the solution φ of (1) depends continuously on the right-hand side f.

A problem is ill-posed if one of i), ii), or iii) does not hold.

Hadamard [8] introduced the notion of well-posed and gave an example of an ill-posed problem via the Cauchy problem for Laplace's equation; e.g.,

$$u_n(x, y) = n^{-2} \exp\{ny\} \sin nx \tag{2}$$

is a solution of the Cauchy problem

$$\begin{aligned} &\frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial y^2} = 0, \qquad -\infty < x < \infty,\ 0 < y, \\ &u_n(x, 0) = n^{-2} \sin nx, \qquad -\infty < x < \infty, \\ &\frac{\partial u_n}{\partial y}(x, 0) = n^{-1} \sin nx, \qquad -\infty < x < \infty. \end{aligned} \tag{3}$$

From the Cauchy-Kowalewski theorem and the Holmgren unicity theory (see I. G. Petrovsky [15]), requirements i) and ii) are satisfied for analytic data. For reasonable norms we see that the data $u_n(x, 0), \partial u_n/\partial y(x, 0) \to 0$ as $n \to \infty$ while $u_n(x, y) \to \infty$ for any $y > 0$.

Another classical ill-posed problem is that of numerical analytic continuation. One can regard the family of functions $\{z^n\}$ as continuations of $f(z) = 0$ outward from a neighborhood of the origin. As analytic continuations exist (modulo singularities) and are unique, i) and ii) are satisfied. However, while $z^n \to 0$ for $|z| < 1$, $|z^n| = 1$ for $|z| = 1$ and $z^n \to \infty$ for $|z| > 1$. Thus, while $f(z) = 0$ can be continued from a neighborhood throughout the entire plane, errors in the values of $f(z)$ within a neighborhood of zero rapidly destroy the significance of any numerical continuation.

The class of problems of the continuation of solutions of partial differential equations from a subset of the domain of definition of the solution to the entire domain of definition of the solution are generally ill-posed. For example, within the domain of definition a solution of the heat equation $\partial u/\partial t = \partial^2 u/\partial x^2$ is analytic in the spatial variable and in certain circumstances in the time variable t. Likewise, solutions of

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Laplace's equation are analytic in each variable inside their domains of definition. Consequently, we find problems of continuation of solutions of the heat equation and Laplace's equation to be ill-posed, and we can expect that similar continuation problems for various partial differential equations may also be ill-posed.

Consider a simple example of (1) above. Namely,

$$f(x) = \int_0^x \varphi(t)\, dt, \qquad 0 \le x \le 1 \tag{4}$$

which has the unique solution

$$\varphi(x) = \frac{df}{dx}(x), \qquad 0 \le x \le 1. \tag{5}$$

As the derivative is an unbounded operator over the class of $C^1(0,1)$ functions (consider $f_n(x) = (\sin n\pi x)/\sqrt{n}$), we see that problems of the form [1] which have unbounded inverses are probably ill-posed. Here, the use of the word "probably" results from the standard approach of achieving continuous dependence on the data via a restriction (sometimes physical) upon the class of functions to which the solution can belong. For the example (4)-(5) above if we assume that $\varphi \in C^1(0,1)$ and $\|\varphi'\|_\infty \le M, M > 0$, then $f \in C^2(0,1)$ and $\|f''\|_\infty \le M$. Thus,

$$\varphi(x) = f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{f''(\xi)h^2}{2} \tag{6}$$

and

$$|\varphi(x)| \le 2h^{-1}\|f\|_\infty + \frac{1}{2}Mh^2, \qquad h > 0, \tag{7}$$

whence it follows from $2h^{-1}\|f\|_\infty = \frac{1}{2}Mh^2$ that

$$\|\varphi\|_\infty \le 4^{2/3} M^{1/3} \|f\|_\infty^{2/3}. \tag{8}$$

Thus, under the above restriction on the solution φ, the continuous dependence of such solutions on the data f with respect to the sup norm $\|f\|_\infty = \sup_{0 \le x \le 1} |f(x)|$ is given by (8).

There is a vast literature on ill-posed problems. As we have seen, it is quite natural for many of these problems to employ function spaces and linear operators which motivate perhaps the rationale for the attempt at lumping these two topics together.

This text consists of material divided into two parts and a supplement entitled "Inversion Formulas in Inverse Problems" by A. L. Bukhgeim. The first part, entitled "Linear Operators", is divided into three chapters: Chapter 1, "Differentiation"; Chapter 2, "Integration"; and Chapter 3, "Linear Operators". These chapters consist of a 225-page, densely packed survey of nearly a hundred years of twentieth-century analysis. With respect to presentation of material, one can find better treatments on differentiation in Deimling [5], Jones [9], Showalter [6], and Yosida [20]; on integration in Jones [9] and Klambauer [10]; and on linear operators in Arkheizer and Glazman [1], Showalter [16], Taylor [17], and Yosida [20].

The second part, entitled "Ill-Posed Problems", consists of 94 pages divided into five chapters: Chapter 4, "Classical Problems"; Chapter 5, "Ill-Posed Problems"; Chapter 6, "Operator and Integral Equations"; Chapter 7, "Evolution Equations"; and Chapter 8, "Problems of Integral Geometry". M. M. Lavrentév gave a better treatment of analytic continuation in his 1967 Springer Tract [12] and for many other topics in ill-posed problems in his monograph with V. G. Romanov and S. P. Sisatskij [13].

For a reader interested in the topic of ill-posed problems, the reviewer recommends that the reader begin with M. M. Lavrentév's 1967 Springer Tract [12] and G. Milton Wing's 1991 *Primer on Integral Equations of the First Kind* [19]. Next the reader should consider Springer Lecture Notes in Mathematics, vol. 316, edited by R. J. Knops [11], and the Society for Industrial and Applied Mathematics's (SIAM) Regional Conference Series in Applied Mathematics, vol. 22, by L. E. Payne [14]. From this point the reader could consider Pitman's Research Notes in Mathematics, vol. 1, edited by A. Carasso and A. P. Stone [3]; Pitman's Research Notes in Mathematics, vol. 105, by G. W. Groetsch [7]; Birkhäuser's ISNM, vol. 77, edited by J. R. Cannon and U. Hornung [2]; Academic Press's Notes and Reports in Mathematics in Science and Engineering, vol. 4, edited by H. W. Engl and C. W. Groetsch [6]; SIAM's Proceedings on *Inverse Problems in Partial Differential Equations*, edited by D. Colton, R. Ewing, and W. Rundell [4]; and Ultrecht: VSP/Moscow: TVP Science Publishers Proceedings on the International Conference, Moscow, August 19–25, 1991, edited by A. N. Tikhonov [18].

Finally, the supplement of the text under review here, entitled "Inversion Formulas in Inverse Problems", treats the connection between tomography and the theory of A-analytic functions and the reconstruction of phonon spectra from the heat capacity data. The specialists in tomography and the physicists interested in phonons may find the respective parts of the supplement of interest.

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