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Generalized topological degree and semilinear equations, by W. V. Petryshyn, Cambridge Tracts in Mathematics, vol. 117, Cambridge University Press, Cambridge, 1995, x+240 pp., $59.95, ISBN 0-521-44474-8

The *topological degree* is a fundamental concept in algebraic topology and in analysis, and the number of its applications to nonlinear differential equations has increased at an impressive rate during the whole second half of this century.

The concept can be rooted in the fundamental work of Kronecker [8] on systems of smooth real functions $f_0, f_1, \ldots, f_n$ of n real variables such that 0 is a regular value for f_0, the set $K = f_0^{-1}(]-\infty, 0])$ is bounded and all f_j do not vanish simultaneously. In this situation, letting $f = (f_1, \ldots, f_n)$, Kronecker showed that the integral (written in modern notations)

$$\chi(f_0, f_1, \ldots, f_n)$$

$$:= \frac{1}{\mathrm{vol}\, S^{n-1}} \int_{\partial K} \sum_{j=1}^{n} (-1)^{j-1} \frac{f_j df_1 \wedge \ldots \wedge df_{j-1} \wedge df_{j+1} \wedge \ldots \wedge df_n}{(f_1^2 + \ldots + f_n^2)^{n/2}}, \tag{1}$$

is equal to the number

$$\sum_{x \in f^{-1}(0) \cap K} \mathrm{sign\ Jac}\ f(x),$$

when this sum makes sense, i.e. when $f^{-1}(0) \cap K$ is finite and the Jacobian Jac f of f does not vanish on $f^{-1}(0) \cap K$. Consequently, the *Kronecker integral* (1) provides an "algebraic count" of the number of zeros of f in K. For $n = 2$, the corresponding integral

$$\frac{1}{2\pi} \int_{\partial K} \frac{f_1 df_2 - f_2 df_1}{f_1^2 + f_2^2}$$

was already considered by Cauchy, when he extended to general planar mappings his index theory for holomorphic functions. Kronecker's work was rooted in the earlier researches of Sturm, Liouville, Cauchy, Hermite and Sylvester on the number of real roots of algebraic equations and in Gauss's pioneering work in potential theory and linking of curves.

The first mathematician to understand fully and exploit Kronecker's theory in various fields of analysis was Poincaré, who explicitly noticed and used the invariance of $\chi(f_0, f_1, \ldots, f_n)$ under a continuous deformation of the f_j such that $(f_0, f_1, \ldots, f_n)$ never vanishes and K remains bounded [14]. In particular, as early as 1883, Poincaré [15] already stated and proved (in his usual sketchy way) an n-dimensional version of the intermediate value theorem known today as *Miranda's theorem* and equivalent to the famous *Brouwer's fixed point theorem*: any continuous mapping g from a compact convex subset C of $\mathbb{R}^n$ into itself has at least one fixed point. Around 1910, Hadamard [6] and Brouwer [2] transformed Kronecker's integral into a topological tool by extending it to continuous mappings f and more general sets K, and showing its importance in topology, through the

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proof of fundamental results on the *invariance of dimension* and on the *invariance of domain.* Kronecker's integral became the *Brouwer degree* $\deg_B[f, \Omega, y]$, defined for any continuous mapping $f : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^n$ such that $y \notin f(\partial\Omega)$, where Ω is open and bounded. The Brouwer degree can easily be extended to continuous mappings between two *oriented* vector spaces of the same finite dimension.

In 1922, motivated by some existence problems for nonlinear differential equations, G.D. Birkhoff and Kellogg [1] extended the Brouwer fixed point theorem to mappings g defined on the *infinite-dimensional spaces* $C([a, b])$ and $L^2(a, b)$, under the assumption that g and the closed bounded convex set C are such that $g(C)$ is relatively compact. Around 1930, Schauder [17] proved this result for *general Banach spaces* and correspondingly generalized Brouwer's invariance of domain theorem to one-to-one mappings which have the form $f = I - g$, where g is *completely continuous,* i.e. is continuous and takes bounded sets into compact sets. The road was paved for the bold step of extending the whole Brouwer degree theory to this class of mappings, and this was Leray and Schauder's achievement in their wonderful paper of 1934 [10], which contains also striking applications to nonlinear integral equations and Dirichlet problems.

Although the importance of Leray-Schauder's paper was immediately understood, its applications and developments were not very numerous until 1950. Theoretical progress was made by Leray himself, Rothe, Tychonoff, Nagumo; and some applications, mostly to fluid mechanics and integral equations, were made by Jacob, Kravtchenko, Dolph and Bers. A nice account was given by Leray [9] at the International Congress of Mathematicians of Cambridge (1950), where references can be found. At this moment, the only existing monographs dealing with Leray-Schauder degree were Nagumo's one in Japanese [13] and C. Miranda's one in Italian [12].

The fifties and the sixties showed an acceleration in the number of applications of Leray-Schauder degree to various problems of existence, multiplicity and bifurcation of solutions of nonlinear equations in Banach spaces and of nonlinear integral or differential equations. One can mention the important and seminal work of M.A. Krasnosel'skii, who initiated the topological approach to *bifurcation theory* [7], the contributions of Ladyzenskaya, Ural'ceva, Gilbarg, Nirenberg and Browder to *nonlinear elliptic equations* (see references in [5]), and the first applications to *ordinary differential equations* starting with Stoppelli (see references in [16]). Those papers did not lead to any modification of the theory with respect to Leray-Schauder's original paper.

If one excepts two rather unnoticed notes of Caccioppoli in 1936 [4] on nonlinear Fredholm mappings of index zero, it was only in the late sixties that various degrees for mappings distinct from compact perturbations of identity in Banach spaces were constructed. For example, Browder and Nussbaum (motivated by Leray-Schauder's treatment of quasilinear elliptic equations) developed a degree for some *intertwined representations of mappings* between Banach spaces, Nussbaum (motivated by Darbo's extension of Schauder fixed point theorem) extended Leray-Schauder degree to *k-set-contractive* perturbations of identity in Banach spaces, and Elworthy and Tromba (motivated by Smale's pioneering work) extended Leray-Schauder theory to some *nonlinear Fredholm mappings between Banach manifolds.*

The same period saw the first paper, by Browder and Petryshyn [3], on a *degree theory for A-proper mappings* between Banach spaces. They were motivated by the recent development of the theory of *monotone* and *accretive operators.* To compare their approach with Leray-Schauder's theory, let us first recall that the Leray-Schauder degree of $I-g$ in Ω over y in a Banach space X is constructed from Brouwer degree by approximating the compact mapping g over Ω by mappings g_ϵ with range in a finite-dimensional subspace X_ϵ (containing y) of X and showing that, if $0 \notin (I-g)(\partial\Omega)$, the values of the Brouwer degrees $\deg_B[(I-g_\epsilon)|_{X_\epsilon}, \Omega \cap X_\epsilon, y]$ stabilize for sufficiently small positive ϵ. Their common value is then the *Leray-Schauder degree* $\deg_{LS}[I-g, \Omega, y]$ of $I-g$ on Ω over y. Thus, in Leray-Schauder's construction, the finite-dimensional approximation depends upon the mapping.

The construction of a degree theory for A-proper mappings between X and Y requires the existence of a suitable admissible *approximation scheme* for the real *separable* Banach spaces X and Y. Let $\{X_n\} \subset X$ and $\{Y_n\} \subset Y$ be sequences of oriented finite dimensional subspaces such that $\dim X_n = \dim Y_n$ and let W_n be a linear map of Y onto Y_n for each $n \in \mathbb{Z}^+$. The scheme $\Gamma_A = \{X_n, Y_n, W_n\}$ is said to be *admissible* for (X, Y) if $\mathrm{dist}(x, X_n) \to 0$ as $n \to \infty$ for each $x \in X$ and $\{W_n\}$ is uniformly bounded. An important special case is the *projectionally complete* scheme $\{X_n, Y_n, Q_n\}$ where Q_n is a linear projection of Y onto Y_n such that $Q_n y \to y$ as $n \to \infty$ for each $y \in Y$. We can now define the concept of *A-proper* map $T : D \subset X \to Y$ with respect to Γ_A, by the properties that $W_n T : D \cap X_n \to Y_n$ is continuous and the following condition holds : if $\{x_{n_j}\}$, with $x_{n_j} \in D \cap X_{n_j}$, is any bounded sequence such that $W_{n_j}(T(x_{n_j}) - g) \to 0$ for some $g \in Y$, then there exists a subsequence $\{x'_{n_j}\}$ and $x \in D$ such that $x'_{n_j} \to x$ in X and $T(x) = g$. A-proper mappings can be naturally associated to some nonlinear boundary value problems and to various classes of monotone-like operators.

If now $D \subset X$ is a dense linear subspace, $G \subset X$ is open, bounded and such that $G_D = G \cap D \neq \emptyset$, and if $T : \overline{G_D} \to Y$ is A-proper with respect to the complete projectional scheme $\Gamma = \{X_n, Y_n, Q_n\}$ and such that $y \notin T(\partial G \cap D)$, the degree $\mathrm{Deg}[T, G_D, y]$ of T on G_D over y is the subset of $\mathbb{Z}' = \mathbb{Z} \cup (+\infty) \cup (-\infty)$ given by:

(a) an integer $m \in \mathrm{Deg}[T, G_D, y]$ provided there exists $\{n_j\} \subset \mathbb{Z}^+$ with $n_j \to \infty$ such that $m = \deg_B[T_{n_j}, G_{n_j}, Q_{n_j} y]$ for all $j \geq 1$;

(b) $+\infty$ (or $-\infty$) $\in \mathrm{Deg}[T, G_D, y]$ if there exists a sequence $\{n_j\} \subset \mathbb{Z}^+$ such that $\lim_{j\to\infty} \deg_B[T_{n_j}, G_{n_j}, Q_{n_j} y] = +\infty$ (or $-\infty$).

As in Leray-Schauder's approach, an approximation by mappings between spaces of the same finite dimension is used, but the approximation scheme, as is the case in *Galerkin method,* is the same for all mappings. One can prove that this new (multivalued) degree conserves the main properties of Leray-Schauder degree.

Most of the applications given in Petryshyn's book are devoted to the case of *semilinear* mappings of the form

$$Tx = Lx - Nx,$$

where $L : D(L) \subset X \to Y$ is an unbounded densely defined linear Fredholm mapping of index 0 and N a suitable nonlinear perturbation. When N is compact with respect to L, a degree theory had been constructed for $L - N$ by the reviewer in 1972 [11], through the introduction of an associated fixed point problem, and many applications to ordinary and partial differential equations have followed in the last twenty years. Petryshyn's degree for A-proper mappings allows one to deal with more general situations, and in particular with differential equations whose nonlinear perturbation may contain derivatives of the same order as the highest one in the linear part.

The rest of the monograph is devoted to the use of degree theory for A-proper mappings in the study of the *periodic boundary value problem* for differential equations of the form

$$x'' - f(t, x, x', x'') = y(t),$$

or

$$(a(t)x^{(n-1)})' + a_1 x^{(n-1)} + f(t, x, x', \dots, x^{(m)}) = y(t), \quad (m \leq n-2),$$

the *Dirichlet boundary value problem* for

$$x^{(4)} + x'' - f(t, x, x', \dots, x^{(4)}) = y(t),$$

some *abstract Landesman-Lazer* problems, and various *boundary value problems for semilinear elliptic partial differential equations.* Besides existence theorems, one also finds a few results on the *bifurcation* of solutions and on the structure and the *covering dimension of the solution set* of some nonlinear equations whose linear part is Fredholm with positive index.

It is well known that the hardest part in the application of any degree technique to some nonlinear differential equation is obtaining *a priori estimates* for the possible solutions of some deformation of the equation to a simpler one. In most of the examples treated in this monograph, those a priori bounds are essentially obtained in the same way as in the simpler situations where more classical degree theories can be used. Therefore, besides an initiation to the theory of degree for A-proper mappings and some of its applications, Petryshyn's monograph presents a wide survey of recent existence results for various semilinear boundary value problems, and constitutes an introduction to the large literature devoted to this class of problems.

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