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HOMOLOGY OF ALGEBRAIC VARIETIES: AN INTRODUCTION TO THE WORKS OF SUSLIN AND VOEVODSKY

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ABSTRACT. We give an overview of the ideas Suslin and Voevodsky have introduced in their works on algebraic cycles and their relation to the mod-n homology of algebraic varieties.

1. INTRODUCTION

The recent series of papers by Suslin, Voevodsky, Suslin-Voevodsky and Friedlander-Voevodsky ([49], [50], [51], [52], [53] and [20]) has developed a remarkable new viewpoint in the study of algebraic cycles. A new "topology" defined by Voevodsky, the qfh-topology, relates algebraic cycles to certain representing sheaves for this topology, and thus allows a systematic application of the powerful methods of sheaf-theory in areas which heretofore have been approached by essentially algebro-geometric or homotopy theoretic means. We hope to give here, not a full overview, but rather a sample of these new techniques and the results they have made possible, concentrating on the applications to the mod-n theory.

2. HISTORICAL BACKGROUND

Let X be a CW-complex. The topological K_0 (see [2], [3]) of X, $K_0^{\mathrm{top}}(X)$, is defined as an abelian group via generators and relations: the generators are the isomorphism classes $[E]$ of complex vector bundles $E \to X$, with relations given by *stable equivalence*:

$$[E] \equiv [E'] \iff E \oplus e^n \cong E' \oplus e^n \text{ for some } n,$$

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where e^n is the trivial rank n bundle on X. For X a finite CW complex, this group is also given as the homotopy classes of maps of X to the classifying space $B\mathrm{GL}_{\mathbb{C}}$ of the topological group

$$\mathrm{GL}_{\mathbb{C}} := \varinjlim_N \mathrm{GL}_{N,\mathbb{C}},$$

where $\mathrm{GL}_{N,\mathbb{C}}$ is the topological group of invertible n by n matrices over $\mathbb{C}$. The higher topological K-theory of a finite CW-complex X can then be defined as the homotopy groups of a function space:

$$K_n^{\mathrm{top}}(X) := \pi_n(\mathrm{Hom}(X, B\mathrm{GL}_{\mathbb{C}})).$$

The filtration of X via its k-skeleta, together with Bott periodicity:

$$K_n^{\mathrm{top}}(X) \cong K_{n+2}^{\mathrm{top}}(X),$$

gives rise to the *Atiyah-Hirzebruch spectral sequence* relating singular cohomology and topological K-theory; this spectral sequence degenerates rationally, giving the isomorphisms

$$K_{\mathrm{even}}^{\mathrm{top}}(X) \otimes \mathbb{Q} \cong \oplus_n H^{2n}(X, \mathbb{Q})$$
$$K_{\mathrm{odd}}^{\mathrm{top}}(X) \otimes \mathbb{Q} \cong \oplus_n H^{2n+1}(X, \mathbb{Q}).$$

The algebraic K_0 of an algebraic variety X is defined as the abelian group with generators the isomorphism classes $[E]$ of algebraic vector bundles E on X, with relations given by setting

$$[E] = [E'] + [E'']$$

if there exists an exact sequence

$$0 \to E' \to E \to E'' \to 0.$$

If X is affine, all such exact sequences split, and one gets the same group by imposing the relation of stable equivalence as in the topological case; but in general, stable equivalence is a weaker relation (they agree, however, in the topological setting). In fact, the algebraic K_0 was defined (by Grothendieck) *before* the topological case, but the higher K-theory in the topological setting was defined before the algebraic case.

Pursuing the analogy with the topological situation, Karoubi and Villamayor [31] gave a definition of higher algebraic K-theory of a ring R by means of the *discrete* group $\mathrm{GL}(R)$, where the topology of $\mathrm{GL}_{\mathbb{C}}$ is replaced by a certain simplicial structure. As this idea is central to our whole discussion, we give a description in a somewhat more general setting.

The algebraic version of homotopy is gotten by replacing the unit interval with the affine line $\mathbb{A}^1$. Following this further, one considers the cosimplicial variety Δ^*, with n-cosimplices Δ^n given as the hyperplane in $n+1$-space defined by the linear equation

$$\sum_{i=0}^n t_i = 1.$$

If one takes the t_i to be real numbers, with $0 \le t_i \le 1$, this is the usual n-simplex; as the usual expressions for the co-face and co-degeneracy maps between the n-simplex and the $n-1$-simplex are linear functions of the t_i, one obtains the structure of a cosimplicial variety on Δ^* by using the same formulas as in the real case.

Starting with the map

$$\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$$
$$(s,t) \mapsto st$$

one can construct a map

$$H\colon \mathbb{A}^1 \times \Delta^* \to (\mathbb{A}^1 \times \Delta^*)^{[0,1]}, \tag{2.1}$$

having the formal properties of a "homotopy" between the identity map on $\mathbb{A}^1 \times \Delta^*$ and the map

$$(x,t) \mapsto (0,t).$$

Now, suppose we have a functor (here **Sch** is the category of schemes)

$$F\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Sets}.$$

We may form the new functor

$$F_h\colon \mathbf{Sch}^{\mathrm{op}} \to \textbf{Simplicial Sets} \tag{2.2}$$

by

$$F_h(X) = F(X \times \Delta^*).$$

The simplicial set $F_h(X)$ then satisfies the *homotopy property*: by applying F to (2.1), one shows that the natural map

$$F_h(X) \to F_h(X \times \mathbb{A}^1)$$

is a homotopy equivalence on the geometric realization. Similarly, if we have a functor

$$F\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Ab},$$

we may form the new functor with values in chain complexes

$$F_{h\mathbb{Z}}\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{C}(\mathbf{Ab}) \tag{2.3}$$

by taking the chain complex associated to the simplicial abelian group F_h. The functor $F_{h\mathbb{Z}}$ is also homotopy invariant: the natural map

$$F_{h\mathbb{Z}}(X) \to F_{h\mathbb{Z}}(X \times \mathbb{A}^1)$$

is a quasi-isomorphism.

Now back to Karoubi-Villamayor K-theory. Suppose X is an affine variety with ring of functions R. The group $\mathrm{GL}_N(R)$ is the group of algebraic maps

$$X \to \mathrm{GL}_N,$$

where GL_N is the open subscheme of affine N^2 space defined as the locus where the determinant function is non-zero. Replacing X with $X \times \Delta^*$ gives the simplicial ring $\Delta_*(R)$ with

$$\Delta_n(R) = R[t_0, \dots, t_n]/(\sum_i t_i - 1)$$
$$\mathrm{Spec}\ \Delta_*(R) = X \times \Delta^*,$$

and the simplicial group $\mathrm{GL}_N(\Delta_*(R))$:

$$\mathrm{GL}_N(\Delta_n(R)) = \mathrm{Hom}_{\mathrm{alg}}(X \times \Delta^n, \mathrm{GL}_N).$$

One then takes the geometric realization $|\mathrm{GL}_N(\Delta_*(R))|$ of this simplicial set, passes to the limit over N, $|\mathrm{GL}(\Delta_*(R))|$, and defines the *Karoubi-Villamayor K-theory of R* by

$$KV_n(R) = \pi_{n-1}(|\mathrm{GL}(\Delta_*(R))|).$$

One can extend this definition to an arbitrary scheme by a sheafification process.

Quillen gave another definition of higher algebraic K-theory, first for rings by using his plus-construction [38], then for arbitrary exact categories via his categorical Q-construction [39]; this definition gained wide acceptance as the "correct" one in the general setting. It turns out that, for a regular scheme X (the algebro-geometric version of a manifold), the Karoubi-Villamayor K-theory agrees with the Quillen K-theory, although for singular schemes, the two definitions are not in general the same.

In contrast with the topological case, the algebraic version of singular cohomology, the so-called "motivic cohomology", was not completely defined until some fifteen years after the definition of algebraic K-theory. The first step towards the full definition of motivic cohomology was, however, taken quite a bit earlier, arising in the 50's with the construction of the Chow ring of algebraic cycles modulo rational equivalence (see e.g. [56], [45], [8], [9], [7], [12] and [25]). What emerged from these constructions was an algebraic homology-like theory built out of the free abelian group on the algebraic subvarieties of X, the *algebraic cycles on X*. Replacing the unit interval with the affine line as above leads to the relation of *rational equivalence*, an algebraic version of homology. More precisely, a cycle is *rationally equivalent to zero* if it is of the form

$$\mathrm{pr}_X(W \cdot (X \times 0) - W \cdot (X \times 1))$$

for W a cycle on $X \times \mathbb{A}^1$, where pr_X is the projection on X, and $\cdot$ is the intersection product (one requires that W contains no component of the form $W_0 \times t$, so that the intersection product is defined). One then defines the *Chow group* of X as the group of algebraic cycles modulo rational equivalence.

For smooth quasi-projective varieties, this homology-like theory has a cohomological flavor as well: the intersection of subvarieties extends to the *intersection product* on the group of cycles mod rational equivalence, defining the *Chow ring*

$$\mathrm{CH}^*(X) := \oplus_q \mathrm{CH}^q(X),$$

with the grading by codimension. In addition, the Chow ring admits functorial pull-back maps

$$f^*\colon \mathrm{CH}^*(X) \to \mathrm{CH}^*(Y)$$

for arbitrary maps $f\colon Y \to X$ between smooth, quasi-projective varieties. Grothendieck's theory of Chern classes with values in the Chow ring [25], together with the Grothendieck-Riemann-Roch theorem [7], gives the isomorphism

$$\sum_q c_q\colon K_0(X) \otimes \mathbb{Q} \cong \oplus_q \mathrm{CH}^q(X) \otimes \mathbb{Q}.$$

The major inadequacy of both the algebraic K_0 and the Chow ring is the lack of good *localization sequence*, relating the theory for X, an open subscheme U, and the closed complement $Z := X \setminus U$. Both theories have the beginning (or end, depending on your point of view) of such a sequence; Quillen's uniform definition of the higher K-groups, together with his localization theorem (see [39]), filled this gap on the K-theory side, giving the long exact sequence

$$\ldots \to K_p(Z) \to K_p(X) \to K_p(U) \to K_{p-1}(Z) \to \ldots \to K_0(X) \to K_0(U) \to 0$$

(this is for X and Z smooth; there is a similar sequence in general).

The first successful definition of motivic cohomology was supplied by Bloch in 1985, with his construction of the *higher* Chow groups [4]. The idea is to make sense of the algebraic cycles on the co-simplicial variety $X \times \Delta^*$. The technical problem here is that an arbitrary codimension q subvariety of $X \times \Delta^p$ may not intersect a face $X \times \Delta^{p'}$ in codimension q, and thus the intersection product with this face is not defined; Bloch solves this by considering the subgroup generated by subvarieties which *do* intersect all faces in the correct codimension. This gives the homological complex $\mathcal{Z}^q(X,*)$, with $\mathcal{Z}^q(X,p)$ being the subgroup of the cycles on $X \times \Delta^p$ just described, and differential given by the pull-back via the coboundary maps in $X \times \Delta^*$ (i.e., intersection with the codimension one faces $X \times \Delta^{p-1}$ of $X \times \Delta^p$). The higher Chow groups are then defined as the homology

$$\mathrm{CH}^q(X,p) := H_p(\mathcal{Z}^q(X,*)).$$

This extends the definition of the Chow ring $\mathrm{CH}^q(X)$, via the identity

$$\mathrm{CH}^q(X) = \mathrm{CH}^q(X,0).$$

It wasn't until around 1992 that the localization property for $\mathrm{CH}^q(X,*)$ was proved (see [5]), and this only for varieties over a field of characteristic 0. We do know, however (see e.g. [32]), that there is a close relation between the higher algebraic K-theory and the higher Chow groups of a variety X which is smooth over a field, so the higher Chow groups do seem to be the "correct" groups for motivic cohomology.

3. The Quillen-Lichtenbaum conjecture

Let X be a variety defined over $\mathbb{C}$, $X(\mathbb{C})$ the topological space of solutions to the equations defining X. Passing from algebraic maps to continuous maps defines the comparison map from algebraic to topological K-theory

$$K_p^{\mathrm{alg}}(X) \to K_p^{\mathrm{top}}(X(\mathbb{C})).$$

In general, this map is far from being an isomorphism. However, the situation seems to get quite a bit simpler if one takes K-theory with mod n coefficients (defined via mod n homotopy groups of the appropriate spaces):

$$K_p^{\mathrm{alg}}(X;\mathbb{Z}/n) \to K_p^{\mathrm{top}}(X(\mathbb{C});\mathbb{Z}/n).$$

The so-called *Quillen-Lichtenbaum conjecture* asserts (in one of its forms) that this map is an isomorphism for $p \geq 2\dim_{\mathbb{C}}(X)$.

One can extend this to varieties over more general fields as follows. Grothendieck *et al.* [26] have defined the *étale cohomology*, $H^p_{\text{ét}}(X,\mathbb{Z}/n)$, of a scheme X, which (for n invertible on X) has many of the formal properties of the mod n singular cohomology of a space (see §5 for more information). In fact, the *comparison theorem* of Artin gives a natural isomorphism between the étale cohomology $H^p_{\text{ét}}(X,\mathbb{Z}/n)$ of a scheme X defined over $\mathbb{C}$, and the singular cohomology $H^p(X(\mathbb{C}),\mathbb{Z}/n)$ of the analytic space of solutions $X(\mathbb{C})$. Building on the methods of Grothendieck, Friedlander [14], [15], and Dwyer-Friedlander [13] constructed an algebro-geometric version of mod n topological K-theory, the *étale K-theory* $K^{\text{ét}}_*(X;\mathbb{Z}/n)$. There is an

Atiyah-Hirzebruch spectral sequence in the étale case as well, relating $H^*_{\text{ét}}(X, \mathbb{Z}/n)$ and $K^{\text{ét}}_*(X; \mathbb{Z}/n)$.

The Quillen-Lichtenbaum conjecture then asserts that the natural map of mod-n algebraic K-theory to mod-n étale K-theory of a variety X over a field k:

$$K^{\text{alg}}_p(X; \mathbb{Z}/n) \to K^{\text{ét}}_p(X; \mathbb{Z}/n)$$

is an isomorphism for $p \geq 2\dim_k(X) + \text{c.d.}(k)$, where c.d.$(k)$ is the mod n cohomological dimension of the Galois group of $\bar{k}/k$. Via the theory of Chern classes to étale cohomology, this can be interpreted as stating that the Chern class maps induce isomorphisms (for F a field)

$$\sum_{2q-p=m} c_{q,p} \colon K_m(F; \mathbb{Z}/n) \to \oplus_{2q-p=m} H^p_{\text{ét}}(F, \mathbb{Z}/n(q)); \quad 2q \geq p,$$

for n prime to "small primes" (depending on m) and prime to the characteristic of F.

We should mention here that the original conjecture of Quillen and Lichtenbaum (see e.g. [34]) concerned itself only with the K-groups and étale cohomology of *number rings*, and that the conjecture stated above is a slight modification of Conjecture 3.9 in [15].

Thus, one hopes for natural isomorphisms,

$$H^p_\mu(X, \mathbb{Z}/n(q)) \to H^p_{\text{ét}}(X, \mathbb{Z}/n(q)); \quad 2q \geq p,\ q \geq \dim_k(X), \tag{3.1}$$

where $H^p_\mu(X, \mathbb{Z}/n(q))$ is the mod-n motivic cohomology of a smooth k-scheme X; at least provisionally, this can be defined via Bloch's cycle complex as

$$H^p_\mu(X, \mathbb{Z}/n(q)) = H_{2q-p}(\mathcal{Z}^q(X, *) \otimes \mathbb{Z}/n).$$

One main result of Suslin and Voevodsky ([49] and [48]) is that, for varieties over an algebraically closed field, such isomorphisms exist.

In fact, the major player in the proof is not Bloch's cycle complex, but a homological version introduced by Suslin (in a 1988 lecture at Luminy). The *Suslin homology* of a scheme X is defined in terms of families of zero cycles on X, parametrized by the co-simplicial scheme Δ^*. There was no serious progress in the study or application of Suslin homology until Voevodsky introduced his qfh-topology and h-topology in 1992; this provided the needed breakthrough by allowing an interpretation of the mod n Suslin cohomology (the dual of Suslin homology) as a cohomology theory arising from sheaves on a Grothendieck site.

4. Relative 0-cycles and Suslin homology of schemes

A pointed topological space $(X, *)$ freely generates a commutative monoid with $*$ acting as identity, the *pointed infinite symmetric product* $Sp^\infty X$, whose points are the finite formal sums $\sum_i x_i$ with $x_i \in X$, modulo the relation

$$n \cdot * + \sum_i x_i \sim \sum_i x_i.$$

If $(X, *)$ is a connected CW complex, the theorem of Dold and Thom [11] shows that $Sp^\infty X$ represents the homology of X, via a natural isomorphism

$$\pi_n(Sp^\infty X) \cong H_n(X; \mathbb{Z}).$$

One can mimick this in the algebraic setting. Rather than attempting to deal with the infinite dimensional scheme $Sp^\infty X$, Suslin considers a functor which is in principle the one represented by the group completion of $Sp^\infty X$. As in §2, one replaces the topology on $Sp^\infty X$ induced by that of X with the simplicial structure induced by Δ^*.

DEFINITION 4.1. Let X and S be k-schemes, with S smooth and irreducible. Define the group $C_0(S;X)$ to be the free abelian group on the subvarieties W of $X \times S$ such that the projection $p_2\colon W \to S$ is finite and surjective (recall that a map $f\colon Y \to Z$ is finite if f is proper and each fiber of f is a finite set). Set

$$C_n(S;X) = C_0(S \times \Delta^n; X).$$

The group $C_0(S;X)$ is covariantly functorial in X and contravariant in S; thus, we may form the complex (see (2.2))

$$C_*(S;X) := C_0(S \times \Delta^*; X) = C_0(S;X)_{h\mathbb{Z}},$$

and the homology

$$H_p^{\text{sing}}(S;X) := H_p(C_*(S;X)).$$

The groups $H_p^{\text{sing}}(S;X)$ are thus covariant in X and contravariant in S.

EXAMPLE 4.2. Take $S = \operatorname{Spec} k$. Then $H_p^{\text{sing}}(k;X)$ is the *Suslin homology of* X. Sending X to $H_p^{\text{sing}}(X;\mathbb{Z}) := H_p^{\text{sing}}(k;X)$ defines the functor

$$H_p^{\text{sing}}(-;\mathbb{Z})\colon \mathbf{Sch}/k \to \mathbf{Ab}.$$

Let Y be a smooth irreducible k-scheme, and define $C_0(X)(Y)$ by

$$C_0(X)(Y) := C_0(Y;X). \tag{4.1}$$

This defines the (contravariant) functor $C_0(X)$ on the category $\mathbf{Sm}/k$ of smooth, finite-type k-schemes.

5. Grothendieck topologies

Grothendieck topologies play a central role in the arguments of Suslin-Voevodsky; we give here a brief overview. For more details see, e.g., [1], [35], [10], and [26].

Grothendieck introduced the notion of a Grothendieck topology, and the associated category of sheaves for the topology, in the construction of étale cohomology, the algebro-geometric replacement of singular cohomology. To form a topology on a set X, one selects a collection of subsets of X, the *open* subsets for the topology, subject to certain axioms. Grothendieck extended this notion by considering the inclusion of an open subset as a special case of a morphism in a category and considering, for a category $\mathcal{C}$, families of morphisms in $\mathcal{C}$

$$f_\alpha\colon U_\alpha \to X,$$

satisfying axioms which generalize the notion of an open cover.

More specifically, for each object U of $\mathcal{C}$, one must define when a family of morphisms

$$\{U_\alpha \to U \mid \alpha \in A\}$$

in $\mathcal{C}$ is a *cover* of U; one requires

1. *stability under base-change*: if

$$\{U_\alpha \to U \mid \alpha \in A\}$$

is a cover of U, and $V \to U$ is a morphism in $\mathcal{C}$, then the fiber product $U_\alpha \times_U V$ exists for each α, and the family

$$\{p_2\colon U_\alpha \times_U V \to V \mid \alpha \in A\}$$

is a cover of V.

2. *stability under composition*: if

$$\{U_\alpha \xrightarrow{f_\alpha} U \mid \alpha \in A\}$$

is a cover of U, and if

$$\{V_\beta \xrightarrow{g_{\alpha\beta}} U_\alpha \mid \beta \in B_\alpha\}$$

is a cover of U_α for each α, then

$$\{V_\beta \xrightarrow{f_\alpha \circ g_{\alpha\beta}} U \mid \beta \in B_\alpha, \alpha \in A\}$$

is a cover of U.

3. each isomorphism

$$U' \to U$$

forms a cover of U.

A category together with a Grothendieck topology is called a *Grothendieck site*.

We have actually defined above the notion of a Grothendieck *pre-topology* on $\mathcal{C}$; this suffices to define the primary objects of interest: presheaves and sheaves. A Grothendieck pre-topology generates a Grothendieck topology, just as a basis for a topology generates a topology, so we will freely confuse the two notions in what follows.

One can view a topology in the usual sense on a set X as a Grothendieck topology, by defining $\mathcal{C}$ to be the category formed by the inclusion maps $V \to U$ among the open subsets of X, and defining the covers of $U \subset X$ to be collections

$$\{U_\alpha \to U\},$$

with

$$\cup_\alpha U_\alpha = U,$$

i.e., open covers of U.

Conversely, if one has a Grothendieck topology on a category $\mathcal{C}$, one can consider the morphisms

$$f\colon V \to U$$

which occur in some cover $\{f_\alpha\colon U_\alpha \to U\}$ as an "open subset" of U. If the category $\mathcal{C}$ has a final object X, and if each morphism $V \to U$ occurs in some cover of U, then one may consider the Grothendieck topology on $\mathcal{C}$ as a "Grothendieck topology on X", and the objects of $\mathcal{C}$ as the opens for the Grothendieck topology.

In the general case, a Grothendieck topology $\mathfrak{T}$ on a category $\mathcal{C}$ induces the Grothendieck topology $\mathfrak{T}_X$ on each object X of $\mathcal{C}$ by forming the category $\mathcal{C}_X$ of maps $U \to X$ which occur in a cover of X, where a map $(U \to X) \to (V \to X)$ is a commutative triangle

$$\begin{array}{ccc} U & \to & V \\ & \searrow \; \swarrow & \\ & X & \end{array}$$

and using the covers in $\mathfrak{T}$ to define the covers in $\mathfrak{T}_X$. The operation of restricting to an object of $\mathcal{C}$ is like taking the induced topology on a subset of a topological space. One can consider a Grothendieck topology on a category $\mathcal{C}$ as being defined by giving a Grothendieck topology on X for each object X of $\mathcal{C}$, with this assignment being natural in X.

Once one has a Grothendieck topology $\mathfrak{T}$ on a category $\mathcal{C}$, one defines a presheaf F (say of abelian groups) for the topology as a contravariant functor from $\mathcal{C}$ to **Ab**. A sheaf is a presheaf F that satisfies the sheaf axiom: if

$$\{i_\alpha \colon U_\alpha \to U\}$$

is a cover of U, then the sequence

$$0 \to F(V) \xrightarrow{\prod_\alpha F(i_\alpha)} \prod_\alpha F(U_\alpha) \xrightarrow{F(p_2)-F(p_1)} \prod_{\alpha,\beta} F(U_\alpha \times_V U_\beta)$$

is exact. The category $\mathrm{Sh}_{\mathfrak{T}}$ of sheaves of abelian groups for the topology $\mathfrak{T}$ is then an abelian category, so one can perform the usual operations of homological algebra in this setting. If X is in $\mathcal{C}$, using the topology $\mathfrak{T}_X$ on $\mathcal{C}_X$, one has the notions of a pre-sheaf or a sheaf on X, giving the category of sheaves on X for the topology $\mathfrak{T}$, $\mathrm{Sh}_{\mathfrak{T}}(X)$.

There is also the operation of sheafifying a presheaf, so one has in particular the "constant" sheaf A_X on X for each abelian group A. As one has the canonical isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}_{\mathfrak{T}}(X)}(\mathbb{Z}_X, \mathcal{F}) \cong \mathcal{F}(X),$$

the groups $\mathrm{Ext}^p(\mathbb{Z}_X, \mathcal{F})$ are the higher derived functors of the global sections functor; one can therefore define the cohomology of a sheaf $\mathcal{F}$ on X for the topology $\mathfrak{T}$ by

$$H^p_{\mathfrak{T}}(X; \mathcal{F}) = \mathrm{Ext}^p_{\mathrm{Sh}_{\mathfrak{T}}(X)}(\mathbb{Z}_X, \mathcal{F}). \tag{5.1}$$

Example 5.1.

1. The *Zariski* topology on a scheme X is a topology in the classical sense, with opens being complements of algebraic subsets F of X. One makes the Zariski topology into a Grothendieck topology as described above.
2. For X a smooth scheme of finite type over an algebraically closed field k, an étale map is a morphism $i \colon U \to X$ whose differential di_u is an isomorphism for all k points $u \in U$. For example, if X is a smooth variety over $\mathbb{C}$, and $f \colon Y \to X$ is *proper*, then f is étale if and only if the map of complex manifolds $f(\mathbb{C}) \colon Y(\mathbb{C}) \to X(\mathbb{C})$ is a covering space (necessarily finite). More generally, if X is still over $\mathbb{C}$, but not necessarily smooth, then a map $f \colon Y \to X$ is étale if and only if, for each point y of Y, there are neighborhoods U of $y \in Y(\mathbb{C})$ and V of $f(y) \in X(\mathbb{C})$ (in the $\mathbb{C}$-topology) such that f gives an isomorphism
$$f(\mathbb{C}) \colon U \to V.$$
We omit the general definition of an étale morphism.

   The étale topology on a scheme X is given by taking the category of "opens of X" to be the étale maps $U \to X$ (of finite type), and saying that a cover of $U \to X$ is a collection of étale maps
$$\{f_\alpha \colon U_\alpha \to U\}$$
such that
$$U = \cup_\alpha f_\alpha(U_\alpha).$$

Using the methods discussed above, one has the étale cohomology of a scheme X with coefficients in an étale sheaf $\mathcal{F}$, $H^*_{\text{ét}}(X,\mathcal{F})$.

Remark 5.2. Let x be a point of a variety X. The *henselization* X^h_x of X at x plays the role, for the étale topology, of a small neighborhood of x in X in the classical topology. The henselization is gotten by taking the inverse limit over the the collection of pointed étale map

$$(Y,y)\to(X,x).$$

For instance, if the base field k is algebraically closed, and x is a smooth k-point of a smooth k-variety X of dimension d, then X^h_x is isomorphic to the henselization $(\mathbb{A}^d)^h_0$ of affine d-space at the origin.

The examples in 5.1 describe the so-called *small* site: giving a Grothendieck topology for a single scheme X; giving a Grothendieck topology on the category $\mathbf{Sch}/k$ of schemes over a fixed base field k defines what is known as a *big* site.

For example, the étale topology on a scheme X is natural in X: if $U\to X$ is étale, and $f\colon Y\to X$ is a morphism of schemes, then $U\times_X Y\to Y$ is étale. This gives us the big étale site on $\mathbf{Sch}/k$. The big Zariski site is defined similarly.

Now, suppose we have a sheaf $\mathcal{F}$ on $\mathbf{Sch}/k$ for some topology $\mathfrak{T}$. For a k-scheme X, one can restrict $\mathcal{F}$ to X to give the sheaf $\mathcal{F}_X$ for the topology $\mathfrak{T}_X$, and the cohomology $H^*(X,\mathcal{F}_X)$, defined as in (5.1) as the Ext groups

$$H^*(X,\mathcal{F}_X)=\mathrm{Ext}^*_{\mathrm{Sh}_{\mathfrak{T}_X}(X)}(\mathbb{Z}_X,\mathcal{F}_X).$$

One can also define the cohomology on X entirely in the category $\mathrm{Sh}_{\mathfrak{T}}/k$ of sheaves on $\mathbf{Sch}/k$ in the following way: Let

$$\mathrm{Hom}_k(-,X)\colon \mathbf{Sch}/k^{\mathrm{op}}\to\mathbf{Sets}$$

be the functor represented by X:

$$Y\mapsto\mathrm{Hom}_k(Y,X).$$

Form the free abelian group on $\mathrm{Hom}_k(-,X)$, giving the presheaf

$$\mathbb{Z}[\mathrm{Hom}_k(-,X)]\colon \mathbf{Sch}/k^{\mathrm{op}}\to\mathbf{Ab}.$$

Let $\mathbb{Z}_{\mathfrak{T}}(X)$ denote the sheafification of $\mathbb{Z}[\mathrm{Hom}_k(-,X)]$ for the topology $\mathfrak{T}$. It follows from the Yoneda lemma that there is a natural isomorphism

$$H^*_{\mathfrak{T}_X}(X,\mathcal{F}_X)=\mathrm{Ext}^*_{\mathrm{Sh}_{\mathfrak{T}_X}(X)}(\mathbb{Z}_X,\mathcal{F}_X)\cong\mathrm{Ext}^*_{\mathrm{Sh}_{\mathfrak{T}}/k}(\mathbb{Z}_{\mathfrak{T}}(X),\mathcal{F}),\tag{5.2}$$

which is the interpretation of sheaf cohomology on X we wanted.

We shall see in the next section that two new topologies, both finer than the étale topology, provide via (5.2) the link between mod n étale cohomology and Suslin homology.

6. The h-topology and the qfh-topology

Voevodsky's qfh-topology allows one to use the machinery of sheaf cohomology in the study of algebraic cycles; the h-topology is perhaps a more natural construction, but the resulting sheaf cohomology is not so obviously related to cycles. As we shall later see, the mod-n cohomologies in the h- and qfh-topologies agree, so the two points of view complement each other.

We fix a base field k, and write $\mathbf{Sch}/k$ for the category of schemes of finite type over k.

Definition 6.1.

1. Let $f\colon X \to Y$ be a map of k-schemes. f is a *topological epimorphism* if f is surjective on points and if Y has the quotient topology; i.e., a subset U of Y is open if and only if $f^{-1}(U)$ is open in X. f is a *universal topological epimorphism* if, for each map of schemes $Z \to Y$, the projection
$$Z \times_Y X \to Z$$
is a topological epimorphism.
2. The *h-topology* on $\mathbf{Sch}/k$ is the Grothendieck topology for which an h-cover of Y is a universal topological epimorphism $X \to Y$.
3. The *qfh-topology* on $\mathbf{Sch}/k$ is the Grothendieck topology for which a qfh-cover of Y is a universal topological epimorphism $X \to Y$ which is quasi-finite over Y (the inverse image of each point of Y is a finite set).

The following structure theorem gives a more concrete idea of the h-topology and qfh-topology.

Theorem 6.2. *i) Let $V \to Y$ be an h-cover. There is a refinement of V,*
$$\begin{array}{ccc} U & \to & V \\ & \searrow \ \swarrow & \\ & Y & \end{array}$$
and a factorization of $U \to Y$ as
$$U = \coprod_i U_i \xrightarrow{j} X \xrightarrow{p} Z \xrightarrow{q} Y \tag{6.1}$$
where p is a finite morphism, q is the blow-up of a closed subscheme of Y, and j is a Zariski open cover. Conversely, each morphism $U \to Y$ which factors as in (6.1) *is an h-cover.*
ii) Let $V \to Y$ be a qfh-cover. There is a refinement of V,
$$\begin{array}{ccc} U & \to & V \\ & \searrow \ \swarrow & \\ & Y & \end{array}$$
and a factorization of $U \to Y$ as
$$U = \coprod_i U_i \xrightarrow{j} X \xrightarrow{p} Y \tag{6.2}$$
where p is a finite morphism and j is a Zariski open cover. Conversely, each morphism $U \to Y$ which factors as in (6.2) *is a qfh-cover.*

Remark 6.3. i) Clearly the h-topology is finer than the qfh-topology. The qfh-topology is finer than the étale topology.
ii) If the characteristic of k is zero, one may use Hironaka's resolution of singularities to show that each h cover of a k-scheme Y has a refinement $U \to Y$ with U smooth over k. In characteristic $p > 0$, the recent work of de Jong [29] gives the same result. This is certainly *not* the case for qfh-covers, even of a smooth k-scheme. This points out an important technical advantage of the h-topology over the qfh-topology: *every* k-scheme is locally smooth in the h-topology, while only 0 and 1-dimensional k-schemes are locally smooth in the qfh-topology.

7. FAMILIES OF 0-CYCLES AND THE qfh-TOPOLOGY

We have seen in (5.2) that the sheafification $\mathbb{Z}_{\mathfrak{T}}(X)$ of the abelian group generated by the representing presheaf $\mathrm{Hom}(-, X)$ plays a central role in understanding sheaf cohomology. We have the "family of 0-cycles on X" functor of example 4.1:

$$C_0(X)\colon \mathbf{Sm}/k^{\mathrm{op}} \to \mathbf{Ab}.$$

The qfh-topology links sheaf cohomology with algebraic cycles via

Theorem 7.1. *Let p be the exponential characteristic of the base field k ($p = char(k)$ if $char(k) > 0$, $p = 1$ if $char(k) = 0$). The functor $C_0(X)[1/p]$ on $\mathbf{Sm}/k$ extends uniquely to a qfh-sheaf (denoted $c_0(X)$) on $\mathbf{Sch}/k$. Moreover, $c_0(X)$ is naturally isomorphic to the "representing" qfh-sheaf $\mathbb{Z}_{qfh}(X)[1/p]$.*

The map relating $\mathbb{Z}_{\mathrm{qfh}}(X)[1/p]$ and $C_0(X)[1/p]$ is gotten by sending a morphism

$$f\colon Y \to X$$

to the transpose of the graph in $X \times Y$. The proof of the first part of Theorem 7.1 can be divided into two steps: the first step consists in extending $C_0(X)[1/p]$ to normal schemes. This is accomplished in [48] by Galois-theoretic methods, and in [50] by a type of limit process, reminiscent of the method of Weil [56] for defining intersection multiplicities. The extension from normal schemes to arbitrary schemes of finite type is then a fairly formal process, using the fact that a reduced scheme of finite type is a direct limit of normal schemes.

Another crucial property of qfh-sheaves is that they admit *transfers*. Recall that a map of schemes $f\colon X \to Y$ is *finite* if f is proper and quasi-finite ($f^{-1}(y)$ is a finite set for each $y \in Y$). For instance, a finite extension of field $K \to L$ defines a finite morphism Spec $L \to$ Spec K. If P is a presheaf on $\mathbf{Sch}/k$, a transfer on P is gotten by giving a map

$$\mathrm{Tr}_{X/S} = \mathrm{Tr}_p\colon P(X) \to P(S),$$

for each finite morphism $p\colon X \to S$ with X and S reduced and irreducible, and S smooth. These maps should be compatible with pull-back in cartesian squares in a sense which we will leave vague. Now, in the qfh-topology, each finite map $X \to S$ is an open cover, and the sheaf axiom shows that, at least if $X \to S$ is Galois with group of automorphisms G, taking the trace

$$\begin{aligned} \mathrm{Tr}\colon & P(X) \to P(X) \\ & x \mapsto \sum_{g \in G} x^g \end{aligned}$$

defines a map $P(X) \to P(S)$. The general case follows with a bit of additional work.

8. THE h-TOPOLOGY AND THE RIGIDITY THEOREM

As the h-topology is finer than the qfh-topology, an h-sheaf also has transfers. The crucial property of sheaves for the h-topology is expressed by the *rigidity theorem*.

The original precursor of the rigidity theorem may be found in Roitman's work on zero-cycles. The work [42] considers the torsion subgroup of the group of zero-cycles modulo rational equivalence on a smooth projective variety X over an algebraically closed field k, and the behavior of this group under the *Albanese mapping*:

$$\alpha_X\colon \mathrm{CH}_0(X) \to \mathrm{Alb}(X).$$

Here $\mathrm{Alb}(X)$ is the Albanese variety of X, a finite dimensional projective algebraic group. Roitman [41], extending the work of Mumford [37] for surfaces, had shown that the map α_X has a "huge" kernel, assuming that X admits a global algebraic p-form with p at least 2 (and that $k = \mathbb{C}$). In contrast with this, Roitman shows in [42] that α_X induces an isomorphism on the torsion subgroups (prime to the characteristic of k).

The main point in his argument is the rigidity result: Suppose we have a family of zero-cycles on X, parametrized by an algebraic curve C, which are n-torsion mod rational equivalence, with n prime to the characteristic of k. Then the family is *constant* mod rational equivalence. The argument is quite simple: Let $C(k)$ be the set of k-points of C and let $\mathrm{CH}_0(X)_n$ be the n-torsion subgroup of $\mathrm{CH}_0(X)$. Sending $x \in C(k)$ to the corresponding 0-cycle gives the map

$$\rho\colon C(k) \to \mathrm{CH}_0(X)_n;$$

one easily shows that ρ extends to a group homomorphism from the k-points of the Jacobian variety

$$J(\rho)\colon J(C)(k) \to \mathrm{CH}_0(X)_n.$$

Since k is algebraically closed, the group $J(C)(k)$ is n-divisible; hence $J(\rho)$ is the zero map. Since $J(C)(k)$ is generated by the differences $x - y$, where x and y are k-points of C, this shows that $\rho(x) = \rho(y)$ for all $x, y \in C(k)$. The fundamental rigidity result, Theorem 8.2 below, is essentially a formal version of this result; various other extensions, with application to K-theory, had been given by Suslin [46], [47], Gillet-Thomason [27] and Gabber [22].

Definition 8.1. A presheaf $\mathcal{F}$ on $\mathbf{Sch}/k$ is called *homotopy invariant* if the map

$$p_1^*\colon \mathcal{F}(X) \to \mathcal{F}(X \times \mathbb{A}^1)$$

is an isomorphism for all X.

Theorem 8.2. *Let $\mathcal{F}$ be presheaf on $\mathbf{Sch}/k$ which*

1. *is homotopy invariant,*
2. *has transfers,*
3. *is n-torsion: $n\mathcal{F}(T) = 0$ for all k-schemes T, where n is prime to the exponential characteristic of k.*

Let x be a smooth point on a k-variety X, X_x^h the henselization of X at x (see Remark 5.2) and let

$$i_x\colon \mathit{Spec}\ k \to X_x^h$$

be the inclusion. Then the map

$$i_0^*\colon \mathcal{F}(X_x^h) \to \mathcal{F}(\mathit{Spec}\ k)$$

is an isomorphism.

It follows from Theorem 8.2 and Remark 6.3 that the h-sheaf associated to a homotopy invariant presheaf with transfers $\mathcal{F}$ and the constant h-sheaf with value $\mathcal{F}(\mathrm{Spec}\ k)$ have the same n-torsion and n-cotorsion, assuming that k is algebraically closed. This implies the fundamental cohomological rigidity theorem

Theorem 8.3. *Let k be an algebraically closed field, and let $\mathcal{F}$ be a homotopy invariant presheaf on* $\mathbf{Sch}/k$ *which admits transfers. Denote by $\tilde{\mathcal{F}}_h$ the sheafification of $\mathcal{F}$ for the h-topology. Then for all n prime to char(k), there is a canonical isomorphism*

$$Ext^*_h(\tilde{\mathcal{F}}_h, \mathbb{Z}/n) \to Ext^*_{\mathbf{Ab}}(\mathcal{F}(Spec\ k), \mathbb{Z}/n).$$

Remark 8.4. The proof in [49] assumes characteristic zero to allow the use of resolution of singularities, which enables one to show that each k-scheme of finite type admits an h-cover which is smooth over k; the result of de Jong noted in Remark 6.3 allows the proof to go through in arbitrary characteristic.

9. Change of topology

Suppose we have two Grothendieck topologies on $\mathbf{Sch}/k$, $\mathfrak{T}$ and $\mathfrak{T}'$, such that $\mathfrak{T}$ is finer than $\mathfrak{T}'$; i.e., each cover in $\mathfrak{T}'$ is a cover in $\mathfrak{T}$.

We may take a sheaf for the topology $\mathfrak{T}'$ and sheafify it for the topology $\mathfrak{T}$, defining the functor

$$i^*\colon \mathrm{Sh}_{\mathfrak{T}'} \to \mathrm{Sh}_{\mathfrak{T}}.$$

For example, we may compare the h-, qfh- and étale topologies,

$$\mathbf{Sch}/k_{\mathrm{h}} \xrightarrow{\alpha} \mathbf{Sch}/k_{\mathrm{qfh}} \xrightarrow{\beta} \mathbf{Sch}/k_{\mathrm{\acute{e}t}}.$$

The main comparison result is

Theorem 9.1. *Let $\mathcal{F}$ be an étale sheaf and let $\mathcal{G}$ be a qfh-sheaf on* $\mathbf{Sch}/k$*. Then*

$$Ext^*_{\acute{e}t}(\mathcal{F}, \mathbb{Z}/n) = Ext^*_{qfh}(\beta^*\mathcal{F}, \mathbb{Z}/n)$$
$$Ext^*_{qfh}(\mathcal{G}, \mathbb{Z}/n) = Ext^*_h(\alpha^*\mathcal{G}, \mathbb{Z}/n).$$

The proof is fairly straightforward: the comparison of the étale and qfh-cohomology relies on the structure of qfh-covers (Theorem 6.2(ii)) and some elementary facts on finite covers of strictly hensel schemes. One then compares the h-cohomology with étale cohomology, using the structure of h-covers (Theorem 6.2(i)) and the Künneth formula for étale cohomology.

10. Suslin cohomology of qfh-sheaves

We now have all the main ingredients needed to relate Suslin homology and étale cohomology: the representability theorem (Theorem 7.1) and the rigidity theorem (Theorem 8.3). Helped along by the comparison theorem (Theorem 9.1), a straightforward spectral sequence argument completes the proof. It is technically useful to work in a somewhat more general setting, working with an arbitrary presheaf rather than the constant presheaf $\mathbb{Z}$.

Let $\mathcal{F}$ be a presheaf on $\mathbf{Sch}/k$, and let X be in $\mathbf{Sch}/k$. Applying $\mathcal{F}$ to the cosimplicial scheme $X \times \Delta^*$ (as in §2) gives the simplicial abelian group $\mathcal{F}(X \times \Delta^*)$, and the associated (homological) complex $\mathcal{F}_*(X)$:

$$\mathcal{F}_n(X) = \mathcal{F}(X \times \Delta^n).$$

This forms the presheaves $\mathcal{F}_n$ on $\mathbf{Sch}/k$, and the complex of presheaves $\mathcal{F}_*$. For an abelian group A, set

$$C_*(\mathcal{F}) := \mathcal{F}_*(\mathrm{Spec}\ k) = \mathcal{F}(\Delta^*)$$
$$H_*^{\mathrm{sing}}(\mathcal{F}, A) := H_*(C_*(\mathcal{F}) \otimes^L A)$$
$$H^*_{\mathrm{sing}}(\mathcal{F}, A) := H^*(\mathrm{RHom}(C_*(\mathcal{F}), A)).$$

Let $\tilde{\mathcal{F}}_{n\ \mathrm{qfh}}$ denote the qfh-sheaf associated to $\mathcal{F}_n$ and $\tilde{\mathcal{F}}_{\mathrm{qfh}}$ the qfh-sheaf associated to $\mathcal{F}$. We also write $C_*(\mathcal{F})$ for the complex of constant sheaves.

Theorem 10.1. *Let k be an algebraically closed field, and let $n > 0$ be prime to char(k). Let $\mathcal{F}$ be a presheaf on $\mathbf{Sch}/k$ which admits transfers. Then there is a canonical isomorphism*

$$H^*_{sing}(\mathcal{F}, \mathbb{Z}/n) \cong Ext^*_{qfh}(\tilde{\mathcal{F}}_{qfh}, \mathbb{Z}/n).$$

Using Theorem 9.1, and the fact that qfh-sheaves admit transfers (§7), Theorem 10.1 implies

Corollary 10.2. *Let $\mathcal{F}$ be a qfh-sheaf on $\mathbf{Sch}/k$, with k algebraically closed. Then there is a canonical isomorphism*

$$H^*_{sing}(\mathcal{F}, \mathbb{Z}/n) \cong Ext^*_{\acute{e}t}(\mathcal{F}, \mathbb{Z}/n)$$

for all n prime to char(k).

Taking $\mathcal{F} = \mathbb{Z}_{\mathrm{qfh}}(X)$ and using Theorem 7.1 and formula (5.2) gives the main result:

Corollary 10.3. *Let X be a scheme of finite type over an algebraically closed field k, and let n be prime to char(k). Then there is a natural isomorphism*

$$H^*_{sing}(X, \mathbb{Z}/n) \cong H^*_{\acute{e}t}(X, \mathbb{Z}/n).$$

The proof of Theorem 10.1 starts by considering the two spectral sequences

$$I_1^{p,q} = \mathrm{Ext}^p_{\mathrm{qfh}}((\tilde{\mathcal{F}}_q)_{\mathrm{qfh}}, \mathbb{Z}/n) \Longrightarrow \mathrm{Ext}^{p-q}((\tilde{\mathcal{F}}_*)_{\mathrm{qfh}}, \mathbb{Z}/n),$$

$$II_2^{p,q} = \mathrm{Ext}^p_{\mathrm{qfh}}(H_q((\tilde{\mathcal{F}}_*)_{\mathrm{qfh}}), \mathbb{Z}/n) \Longrightarrow \mathrm{Ext}^{p-q}((\tilde{\mathcal{F}}_*)_{\mathrm{qfh}}, \mathbb{Z}/n).$$

The comparison of qfh-cohomology and étale cohomology (Theorem 9.1), and the homotopy invariance of étale cohomology

$$H^*_{\acute{e}t}(X, \mathbb{Z}/n) \cong H^*_{\acute{e}t}(X \times \mathbb{A}^1, \mathbb{Z}/n)$$

lead to a proof that the projections $X \times \Delta^p \to X$ induce isomorphisms

$$\mathrm{Ext}^p_{\mathrm{qfh}}(\tilde{\mathcal{F}}_{\mathrm{qfh}}, \mathbb{Z}/n) \cong \mathrm{Ext}^p_{\mathrm{qfh}}((\tilde{\mathcal{F}}_q)_{\mathrm{qfh}}, \mathbb{Z}/n).$$

This implies the degeneration of the spectral sequence I at E_1, giving the isomorphism

$$\mathrm{Ext}^*_{\mathrm{qfh}}(\tilde{\mathcal{F}}_{\mathrm{qfh}}, \mathbb{Z}/n) \cong \mathrm{Ext}^*_{\mathrm{qfh}}((\tilde{\mathcal{F}}_*)_{\mathrm{qfh}}, \mathbb{Z}/n).$$

By the homotopy invariance of the functors (2.3), the homology presheaf

$$H_q((\tilde{\mathcal{F}}_*)_{\mathrm{qfh}})$$

forms a homotopy invariant presheaf on $\mathbf{Sch}/k$; the transfers for the qfh-sheaves $(\tilde{\mathcal{F}}_q)_{\mathrm{qfh}}$ give transfers for $H_q((\tilde{\mathcal{F}}_*)_{\mathrm{qfh}})$. This allows one to use the comparison with h-cohomology (Theorem 9.1) and the rigidity theorem (Theorem 8.3) to compare the spectral sequence II with the spectral sequence

$$III_2^{p,q} = \mathrm{Ext}^p_{\mathbf{Ab}}(H_q(\mathcal{F}_*(\mathrm{Spec}\ k)), \mathbb{Z}/n) \Longrightarrow \mathrm{Ext}^{p-q}_{\mathbf{Ab}}(\mathcal{F}_*(\mathrm{Spec}\ k), \mathbb{Z}/n),$$

giving the isomorphism

$$\mathrm{Ext}^*_{\mathbf{Ab}}(\mathcal{F}_*(\mathrm{Spec}\ k), \mathbb{Z}/n) \cong \mathrm{Ext}^*_{\mathrm{qfh}}((\tilde{\mathcal{F}}_*)_{\mathrm{qfh}}, \mathbb{Z}/n)$$

and completing the proof.

11. Suslin homology and Bloch's higher Chow groups

To complete the discussion, it is necessary to relate Suslin homology and Bloch's higher Chow groups.

For X in $\mathbf{Sch}/k$, and Y normal, let $Z_0(X)(Y)$ denote the free abelian group on the subvarieties W of $X \times Y$ which are *quasi-finite* over Y. As in §4, sending a smooth Y to $Z_0(X)(Y)$ defines a contravariant functor on $\mathbf{Sm}/k$. Taking Y to be the cosimplicial scheme Δ^* and taking the associated homological complex define the complex $Z_*(X)$, with

$$Z_n(X) = Z_0(X)(\Delta^n).$$

Suppose X has pure dimension d over k. As a subvariety W of $X \times \Delta^p$ which is quasi-finite over Δ^p obviously has the proper intersection with all faces, there is the inclusion of complexes

$$Z_*(X) \to \mathcal{Z}^d(X, *). \tag{11.1}$$

Suslin [48] has shown

Theorem 11.1. *Suppose that X is affine. Then the inclusion* (11.1) *is a quasi-isomorphism.*

Let $\mathrm{CH}^q(X, p; \mathbb{Z}/n)$ denote the homology

$$\mathrm{CH}^q(X, p; \mathbb{Z}/n) = H_p(\mathcal{Z}^q(X, *) \otimes \mathbb{Z}/n) = H_p((\mathcal{Z}^q(X, *) \otimes^L \mathbb{Z}/n).$$

There is an analog of Theorem 7.1:

Theorem 11.2. *There is a unique extension of the functor $Z_0(X)[1/p]$ on $\mathbf{Sm}/k$ to a qfh-sheaf $z_0(X)$ on $\mathbf{Sch}/k$.*

Applying Theorem 10.1 gives

Proposition 11.3. *Let n be prime to char(k) and suppose X is affine of dimension d over k. There is a natural isomorphism*

$$CH^d(X, *; \mathbb{Z}/n)^\vee = H_*^{sing}(z_0(X), \mathbb{Z}/n)^\vee \cong Ext^*_{qfh}(z_0(X), \mathbb{Z}/n).$$

Here $M^\vee$ is the dual

$$M^\vee := \mathrm{Hom}_{\mathbb{Z}/n}(M, \mathbb{Z}/n)$$

of a $\mathbb{Z}/n$-module M.

The qfh-sheaves $z_0(X)$ are contravariantly functorial in X for open immersions, and covariantly functorial for proper maps. If Y is a closed subscheme of X with complement U, the following sequence

$$0 \to z_0(Y) \to z_0(X) \to z_0(U) \tag{11.2}$$

is easily seen to be exact. Letting $\tilde{z}_0(X)_\mathrm{h}$ denote the h-sheaf associated to $z_0(X)$, there is the

Lemma 11.4. *The sequence*

$$0 \to \tilde{z}_0(Y)_h \to \tilde{z}_0(X)_h \to \tilde{z}_0(U)_h \to 0 \tag{11.3}$$

induced from the sequence (11.2) *is an exact sequence of h-sheaves on* $\mathbf{Sch}/k$.

REMARK 11.5. As mentioned in §4, the Suslin homology of the functor $C_0(X)$ is an algebraic version of homology; the Suslin homology of the functor $Z_0(X)$ can similarly be viewed as an algebraic version of *Borel-Moore homology*. In fact, recall from Theorem 7.1 that the functor $C_0(X)[1/p]$ extends to the qfh-sheaf $c_0(X)$ on $\mathbf{Sch}/k$. The h-sheaf $\tilde{c_0}(-)_{\mathrm{h}}$ associated to the qfh-sheaf $c_0(-)$ satisfies a Mayer-Vietoris property, giving the exact sequence

$$(11.4) \qquad 0 \to \tilde{c_0}(U \cap V)_{\mathrm{h}} \to \tilde{c_0}(U)_{\mathrm{h}} \oplus \tilde{c_0}(V)_{\mathrm{h}} \to \tilde{c_0}(U \cup V)_{\mathrm{h}} \to 0.$$

By Theorem 10.1, the sequences (11.3) and (11.4) give rise to a localization sequence for the mod n Suslin homology $H_*^{\mathrm{sing}}(Z_0(-), \mathbb{Z}/n)$ and a Mayer-Vietoris sequence for the mod n Suslin homology $H_*^{\mathrm{sing}}(-, \mathbb{Z}/n)$ (assuming n prime to the characteristic). In [50], modifications of the sequences (11.3) and (11.4) give rise to a localization sequence for $H_*^{\mathrm{sing}}(Z_0(-), \mathbb{Z})$ of $Z_0(-)$ and a Mayer-Vietoris sequence for $H_*^{\mathrm{sing}}(-, \mathbb{Z})$ (assuming characterstic zero).

If X is proper over k, we have

$$z_0(X) = c_0(X).$$

By Theorem 9.1, formula (5.2) and Corollary 10.2, we have the isomorphisms

$$(11.5) \qquad \mathrm{Ext}^*_{\mathrm{h}}(\tilde{z_0}(X)_{\mathrm{h}}, \mathbb{Z}/n) \cong H^*_{\text{ét}}(X, \mathbb{Z}/n)$$

for X proper. Using the long exact Ext-sequence arising from the exact sequence of Lemma 11.4, and the corresponding long exact Gysin sequence for étale cohomology with compact supports, one extends the isomorphism (11.6) to the isomorphism

$$(11.6) \qquad \mathrm{Ext}^*_{\mathrm{h}}(\tilde{z_0}(X)_{\mathrm{h}}, \mathbb{Z}/n) \cong H^*_c(X, \mathbb{Z}/n)$$

to arbitrary X; here H^*_c is the étale cohomology with compact supports.

With the help of (11.6), one arrives at the following comparison of the mod-n higher Chow groups, and étale cohomology for affine X:

Theorem 11.6. *Let X be an affine variety over an algebraically closed field k, and let $n > 0$ be an integer prime to char(k). Assume that $q > d := \dim(X)$. Then there is a natural isomorphism*

$$CH^q(X, p; \mathbb{Z}/n) \cong H_c^{2(d-q)+p}(X, \mathbb{Z}/n(d-q))^\vee.$$

The proof uses the homotopy property to replace X with $X \times \mathbb{A}^{q-d}$, which reduces to the case $q = d$; one then applies Proposition 11.3 and the isomorphism (11.6). Using Poincaré duality, Theorem 8.3 implies

Corollary 11.7. *Let X be a smooth affine variety over an algebraically closed field k, and let $n > 0$ be an integer prime to char(k). Assume that $q \geq d := \dim(X)$. Then there is a natural isomorphism*

$$CH^q(X, p; \mathbb{Z}/n) \cong H_{\acute{e}t}^{2d-p}(X, \mathbb{Z}/n(q)).$$

Theorem 11.6 (resp. Corollary 11.7) extends to quasi-projective X (resp. smooth quasi-projective X) if char$(k) = 0$, using the localization property for the higher Chow groups [5] and the analogous Gysin sequence for étale cohomology with compact supports.

12. Epilogue

We have not described the beautiful constructions and results in the fundamental papers [50], [52] and [20], which form the foundations for many of the constructions we have described. We omitted as well mention of the works on Lawson homology and related topics ([17], [18], and [16]), which formed a large part of the motivation for many of the constructions of [50] and [20]. We have also not described the various constructions of triangulated motivic categories and motivic complexes, in [23], [28], [34], [36], [53] and [33]. Presumably, the categorical constructions all lead to equivalent categories and the complexes all yield equivalent cohomologies, but this is at present not known.

More recently, Suslin and Voevodsky [51] have reduced the Quillen-Lichtenbaum conjectures for motivic cohomology to the *Bloch-Kato conjecture*, which asserts that the *Galois symbol* on Milnor K-theory of a field F defines an isomorphism

$$K_q^M(F)/n \to H^q_{\text{ét}}(F, \mu_n^{\otimes q})$$

for all n prime to the characteristic of F. Relying on results of Rost [43], [44], Voevodsky has given a proof of the mod 2 Bloch-Kato conjecture (see [54]), which thus completely solves the mod 2 Quillen-Lichtenbaum conjecture for motivic cohomology. Using the Bloch-Lichtenbaum spectral sequence [6] relating K-theory and motivic cohomology, Weibel [55] has shown how to compute the mod 2 K-theory of $\mathbb{Z}$; this has recently been extended by Kahn [30] to a computation of the mod 2 K-theory of all number rings, verifying that mod 2 algebraic K-theory agrees with mod 2 étale K-theory for all number rings. There is a whole host of unanswered questions in the area of motivic cohomology and algebraic K-theory; the new viewpoint offered by Voevodsky, Suslin and others will surely lead to further breakthroughs.

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