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3-*Transposition groups*, by Michael Aschbacher, Cambridge University Press, 1997, 260+vii pp., $49.94, ISBN 0-521-57196-0

We begin with a quote from Daniel Gorenstein, the prime force behind the classification of finite simple groups:

> [12, p. 22] To the nonexpert, the name Bernd Fischer is known solely for its connection with a number of sporadic simple groups, but to the practitioner, he is recognized as the founder of *internal geometric analysis* ... a fundamental general technique for studying simple groups, which can reasonably be regarded as second in importance for the classification only to local group-theoretic analysis.
>
> In contrast to local analysis, Fischer's work appears as a personal creation, and aspects of its subsequent development have an almost magical quality. ... in Fischer's case, originality begins with the very question he raised:
>
> > *Which finite groups can be generated by a conjugacy class of involutions, the product of any two of which has order* 1, 2, *or* 3*?*

Here an involution is an element of order 2, and a class as described is called a conjugacy class of 3-transpositions. The group it generates is then a 3-transposition group. Aschbacher's excellent book presents a proof of Fischer's elegant and remarkable theorem of classification and goes on to describe many properties of the groups that arise, particularly the three sporadic Fischer groups. The book is broken into three parts. The first presents Fischer's classification, the second deals with existence and uniqueness issues for the sporadic Fischer groups, and the third details their local structure.

The first one hundred pages of the book present an essentially complete proof of Fischer's theorem, which classified all finite 3-transposition groups $G$ with simple derived subgroup $G'$ and trivial center ($Z(G) = 1$). The prerequisites are minimal — a basic algebra course plus some material from Aschbacher's text [4] (most of what is needed being repeated here in a preliminary chapter). It is remarkable that, in the field of finite group theory where proofs are famous for being long and complicated, this pivotal result can be given an elementary proof in a moderate amount of space. Even more surprising is that, although the theorem was announced nearly thirty years ago, Aschbacher's book is the only place to find a complete proof, published in one spot. Fischer's original version appeared as mimeographed lecture notes printed by Warwick University in the early 1970's, and later he published only the first part of his argument [11]. There have been subsequent treatments, most notably a succession of papers written by Richard Weiss in collaboration with others and culminating in [20]. M.-M. Virotte-Ducharme [18] and F. Zara [21] returned to and extended Fischer's work in their largely unpublished theses. (Related work has been done by H. Cuypers and the reviewer [8].)

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Fischer first came to this problem through his work in the early 1960's on finite distributive quasigroups — finite sets endowed with a binary multiplication for which the right and left multiplication maps are not just permutations (the defining property for a quasigroup) but actual quasigroup automorphisms. His main result [9, Hauptsatz] was that the group generated by the conjugacy class of automorphisms composed of right multiplications is solvable. In his proof, consideration of a minimal counterexample led to the case where all the multiplication mappings have order 2 and all pairs have product of prime power order, for some fixed prime $p$. For such a conjugacy class $D$ of maps and the finite group $G = \langle D \rangle$ (the angle brackets indicate that $G$ is generated by $D$), he was able to prove that $G$ has a normal $p$-subgroup of index 2 [9, Satz 4.1]. (At roughly the same time, Glauberman was considering similar questions for loops [13] and so was led to his important $Z^*$-theorem [14].) The case $p = 3$ was the most difficult.

Fischer next turned to a related problem. Suppose each involution of the conjugacy class $D$ in a finite group is allowed to commute with exactly one other of its conjugates and have product of $p$-power order with all the rest. The conclusion [10] is that $p$ equals 3, the group generated is a symmetric group of permutations on 4 or 5 letters, and the class of involutions is the transposition class — those permutations which exchange two of the letters and leave the others fixed.

The final step was then natural. Fischer considered a conjugacy class $D$ of involutions in a group such that pairs of elements from the class either commute or have product of order 3. The obvious example is the transposition class of a symmetric group, $S_n$, acting on a set of arbitrary size $n$. In deference to the symmetric examples, the class $D$ is called a class of 3-transpositions. The group generated, $G = \langle D \rangle$, is a 3-transposition group. (This definition is the same as that given above — two involutions are equal if and only if their product has order 1, and distinct involutions commute precisely when their product has order 2.)

Initially Fischer suspected that symmetric groups would provide the only examples of 3-transposition groups, but Roger Carter pointed out that this is not so. Indeed, the reflection class of any finite Weyl group with simply laced diagram is a 3-transposition class, the groups of type $A_n$ providing the symmetric examples again. In particular, the Weyl group $W(E_6)$, a six-dimensional orthogonal group over the field with two elements, $GF(2)$, is a 3-transposition group that is not symmetric. This suggests, as is the case, that the class of transvections in any orthogonal group over $GF(2)$ is a 3-transposition class. The same is true of the transvection classes in symplectic groups over $GF(2)$ and unitary groups over $GF(4)$. The Weyl group $W(E_6)$ is also an orthogonal group of dimension 5 over $GF(3)$, and the reflection classes of orthogonal groups over $GF(3)$ provide the last of the four non-symmetric, classical families of 3-transposition classes. Fischer's theorem states that, for a finite 3-transposition group with simple derived subgroup $G'$ and trivial center $Z(G) = 1$, these five classes provide the only examples, except for three exceptions, which Fischer denoted $M(22)$, $M(23)$, and $M(24)$. Of these, $M(22)$ and $M(23)$ are simple, while $M(24)$ has as a subgroup of index 2 its simple derived subgroup $M(24)'$, also called $F_{24}$ in Aschbacher's book. These three simple groups are sporadic, in the sense that they are not members of the natural infinite families of finite simple groups — the alternating groups, the classical groups, and the exceptional groups of Lie type.

The proof of Fischer's theorem, as presented by Aschbacher, in general follows Fischer's original outline. The approach can be thought of as a microcosm of the general classification of finite simple groups, proceeding by induction first to identify the structure of an involution centralizer and then to characterize each group by that data. (Here the centralizer of an element $x$ in the group $A$ is the subgroup of all elements which commute with $x$ and is denoted $C_A(x)$.) For a group $G$ generated by the 3-transposition class $D$, two basic observations are crucial — any non-trivial quotient of $G$ is again a 3-transposition group, and, for any subgroup $H$ of $G$, the set $H \cap D$ is a union of 3-transposition classes in $H$.

Assume, as in Fischer's theorem, that $G$ has trivial center, that $G'$ is simple, and that $G$ is generated by the 3-transposition class $D$. The centralizer $H = C_G(d)$ of the 3-transposition $d$ from $D$ is a proper subgroup of $G$, so induction allows the possibility of identifying the subgroup $\langle H \cap D \rangle$ and reconstructing $G$ from it. The problem with this approach is that the number of initial possibilities for $\langle H \cap D \rangle$ is unmanageable.

The most fundamental difficulty is that $H \cap D$ might not be a single conjugacy class in $H$ but might be the disjoint union of several $H$-classes. Fischer responded to that with his inspired

**Transitivity Theorem.** *Let $G$ be a finite group generated by the 3-transposition class $D$ with $G'$ simple and $Z(G) = 1$. Then, for each $d \in D$, the subgroup $C_G(d)$ (and even its subgroup $\langle D \cap C_G(d) \rangle$ ) is transitive on each of the sets*

$$D_d = \{e \in D \mid \text{ the order of } de \text{ is } 2\}$$
$$A_d = \{e \in D \mid \text{ the order of } de \text{ is } 3\}.$$

Here the permutation action of $G$ and its subgroups on $D$ is the natural conjugation action. Thus $G$ is transitive on the class $D$, and transitivity on $D_d$ is equivalent to the statement that $D_d$ is a single conjugacy class of $C_G(d)$.

Fischer introduced a geometric setting for the Transitivity Theorem. Let $\mathcal{D}(D)$ be the commuting graph of the set $D$ — the graph with vertex set $D$ and edges connecting those pairs of distinct vertices that commute. Then $D_d$ is the set of neighbors of $d$, and $A_d$ is the set of non-neighbors. The Transitivity Theorem says that $G$ acts on $\mathcal{D}(D)$ as a group of automorphisms that is transitive on vertices, edges, and non-edges. Fischer went on to define in a purely combinatorial manner a class of graphs called triple graphs to which these commuting graphs belong and whose members, conversely, always come from 3-transposition groups as in the theorem.

At this point the classification can begin in earnest. For the purpose of this review, we introduce a piece of terminology. Let $H$ be a 3-transposition group with trivial center and $L$ a subgroup of $H$ generated by 3-transpositions of $H$. We say that the 3-transposition group $G$ with 3-transposition class $D$ is of local type $\mathcal{F}(H, L)$ if, for some pair $d, e \in D$ with $de$ of order 3, we have $\langle D_d \rangle / Z(\langle D_d \rangle)$ isomorphic to $H$ by an isomorphism which takes $\langle D_d \cap D_e \rangle$ to $L$. If we do not wish to specifiy $L$, then we merely say that $G$ has local type $\mathcal{F}(H)$. (Notice that, under the Transitivity Theorem, the actual choices of $d$ and $e$ do not affect the local type.) A crucial point in the proof of Fischer's theorem is the following observation, whose spirit can be found in Fischer's Warwick notes, made precise by Weiss [20].

> **The Uniqueness Principle.** For any pair $H$ and $L$, there is (up to isomorphism) at most one 3-transposition group $G$ of local type $\mathcal{F}(H, L)$ with trivial center and $G'$ simple.

As mentioned before, the candidate groups $H$ are identified by induction. The job is to show that suitable $L$ can be found only when there is a known group with that local type. Then the Uniqueness Principle says that the only possible conclusions are those already known, proving the theorem.

We discuss an example, starting with an easy, combinatorial lemma.

> **Lemma.** *Let $L$ be a subgroup of the symmetric group $S_n$, for $n \geq 6$, that is generated by transpositions and such that, for any transposition $f$ in $L$, the transpositions in $L$ which are different from but commute with $f$ generate a subgroup $S_{n-3}$. Then either $L$ is a subgroup $S_{n-1}$ or $n = 6$ and $L$ is a subgroup $S_3 \times S_3$.*

Consider now a 3-transposition group $G = \langle D \rangle$ of local type $\mathcal{F}(S_n, L)$ for some $n \geq 6$. What do we know of $L$? It is $\langle D_d \cap D_e \rangle$ for a pair $d, e$ with product of order 3. Choose a 3-transposition $f$ of $L$. Then by assumption $\langle D_f \rangle$ is isomorphic to $S_n$ and contains $d$ and $e$. Therefore $\langle (D_d \cap D_e) \cap D_f \rangle$ can be calculated within $S_n$ as the common centralizer of two non-commuting transpositions, that is, a subgroup $S_{n-3}$. We conclude that, as a subgroup of $\langle D_d \rangle$ (also isomorphic to $S_n$), the subgroup $L$ satisfies the hypotheses of the lemma and so is either $S_{n-1}$ or is $S_3 \times S_3$ for $n = 6$. Now we note that $S_{n+2}$ has local type $\mathcal{F}(S_n, S_{n-1})$ while $W(E_6)$ has local type $\mathcal{F}(S_6, S_3 \times S_3)$. From the Uniqueness Principle, we conclude:

> **Theorem.** *Let $G$ be a 3-transposition group with trivial center, $G'$ simple, and of local type $\mathcal{F}(S_n)$ with $n \geq 6$. Then $G$ is isomorphic either to $S_{n+2}$ or to $W(E_6)$ for $n = 6$.*

This is the kind of result that finite group theorists love; in the process of proving the natural characterization theorem for the symmetric groups, the anomaly $W(E_6)$ forces itself upon us. There is of course an explanation. The group $S_6$ is isomorphic to the symplectic group $Sp_4(2)$ and so is a transvection centralizer in both of the orthogonal groups $O_6^+(2)$ (isomorphic to $S_8$) and $O_6^-(2)$ (isomorphic to $W(E_6)$). (As an exercise, the reader might complete the argument for this particular theorem using only Weyl groups and the Transitivity Theorem but not the Uniqueness Principle.)

The general proof proceeds by considering groups of local type $\mathcal{F}(H)$ for the various possibilities $H$. It is not possible to assume that $H'$ is simple (the most favorable inductive situation), but Fischer proved that, failing that, $H$ either has a non-central normal 3-subgroup or a non-central normal 2-subgroup. A brief argument proves that the first case leads only to $S_5$. The second case is more serious, since it actually arises for the symplectic groups over $GF(2)$ and the unitary groups over $GF(4)$. There it is possible to recognize the underlying "polar space" geometry within the commuting graph. Each transvection is associated with a singular 1-space of the geometry, and the singular 2-spaces are constructed as specific cliques (complete subgraphs) of the commuting graph. This is of considerable help in setting up the appropriate application of the Uniqueness Principle that gives rise to these groups.

Once those $H$ with non-central normal 2- and 3-subgroups have been handled, it remains to consider the groups in the conclusion to Fischer's theorem as candidates for $H$. This machine works very effectively as we move through the classical groups, but again there is one anomalous case, which occurs when $H$ is $U_6(2)$, a unitary group in dimension 6 over $GF(4)$.

> **Theorem.** *Let $G$ be a 3-transposition group with trivial center, $G'$ simple, and of local type $\mathcal{F}(U_6(2))$. Up to isomorphism there is at most one such $G$. It is simple and not a classical group.*

In fact, $G$ is the first sporadic Fischer group $M(22)$. Once we have found evidence for $M(22)$, we must consider local type $\mathcal{F}(M(22))$ from which emerges at most one group, namely $M(23)$. Similarly $\mathcal{F}(M(23))$ leads to at most one group $M(24)$, while $\mathcal{F}(M(24))$ leads nowhere at all.

At this point, all leads have been exhausted, and the elegant and elementary proof of Fischer's Theorem presented by Aschbacher is complete. The proof occupies the first five chapters of the book. Anyone who wishes to see in action many of the techniques and much of the spirit of the classification of finite simple groups can find no better introduction than these chapters. In particular, they would be suitable and stimulating for an intermediate graduate-level course.

What of Gorenstein's remarks? Certainly Fischer's theorem is important for its location of three of the twenty-six finite sporadic simple groups. But an equally important consequence was a change in the psychology of finite group theory. "Internal geometric analysis" is basically a point of view under which some part of the subgroup structure of a group is viewed as a geometry upon which the group acts as a group of automorphisms. Typically, the objects of the geometry are conjugacy classes or cosets of certain subgroups with incidence determined by a specific group theoretic relationship. The commuting graph of a 3-transposition class is a prime example. Other group theorists had previously thought along these lines, most notably Brauer [6] and Suzuki [15], whose involution centralizer characterizations of $PSL_3(q)$ proceeded by reconstructing internally the projective plane upon which each group acts. Although Brauer did not publish his results until the mid 1960's, he announced a version in his address at the 1954 International Congress in Amsterdam.[1] He particularly remarked upon the geometric nature of his proof. Motivated by this, Brauer defined (in the language of metric spaces) the commuting graph on the non-identity elements of any group. He stated that it (or its subgraph of involutions) would be important "as a kind of geometry in which the fundamental group is given by" the group itself, acting via conjugation.

Brauer and Suzuki's proofs were primarily character theoretic, with the geometric arguments only appearing late and at a point where the group order was already known. In contrast, commuting graphs and triple graphs were at the focal point of Fischer's arguments. The fact that he was able to draw profound results out of an internally created geometry as elementary as the commuting graph was compelling. The impact was particularly dramatic on young group theorists of the early 1970's. Aschbacher's first significant work in group theory [1] was his extension of Fischer's results to "odd transpositions". Aschbacher's critical Standard Form Theorem and its proof [2] have their roots in Fischer's Transitivity Theorem. Fischer's student Franz Timmesfeld also extended Fischer's results considerably by concentrating on

[1] This address was the starting point of modern finite simple group theory. Brauer's report began, "The theory of groups of finite order has been rather in a state of stagnation in recent years." Then at the bottom of page 3 he posed the defining and vitalizing problem: "Given a group $N$ containing an involution $j$ in its center. What are the groups $G$ containing $N$ as a subgroup such that $N$ is the normalizer of $j$ in $G$?"

the symplectic and unitary examples, where the 3-transpositions are in fact long root elements in the standard Lie theoretic setting. Timmesfeld has gone on to develop an extensive theory of "abstract" root elements and subgroups where, starting with a small number of elementary and elegant axioms, large numbers of Lie type and related groups (even infinite ones) can be characterized [17].

The work of Aschbacher and Timmesfeld (and others) demonstrated that many already common techniques, such as weak closure methods and TI-sets, are fundamentally geometric in nature. This viewpoint is particularly evident in their work on the $O_2$-extraspecial problem [3], [16]. The metaphor of internal geometry for subgroup structure has become dominant. It illuminates older methods and lies at the heart of many recent techniques in finite groups such as the amalgam method, $p$-local geometry, and the simplicial approach to uniqueness. This last plays a role in Part II, where it is used in the study of the commuting graph for $M(23)$ and $M(24)$.

Part I of Aschbacher's book closes with Chapter 6, entitled "Beyond Fischer's Theorem". Here Aschbacher discusses related matters such as Francis Buekenhout's elegant geometric axioms for "Fischer spaces" (another case where the original work is unpublished). Aschbacher presents an early result of M. Hall, Jr., characterizing a certain Steiner triple system of order 81 that is related to the quasigroup and loop origins of the 3-transposition problem. (As an illustration of interaction among the various parts, this result of stand-alone geometric interest is also used to prove certain basic 3-transposition properties in Chapter 3 and to construct a 3-local subgroup of $M(23)$ in Chapter 14.) Chapter 6 additionally discusses some of the extensions to Fischer's theorem, particularly the work of Timmesfeld on abstract root elements mentioned above. Also discussed is the work of H. Cuypers and the reviewer culminating in [8], which classifies general 3-transposition groups with trivial center and requires neither finiteness nor simplicity for $G'$. (This of course includes the finite groups of Fischer's theorem, but several of the most difficult cases are handled only by quoting arguments of Fischer, Weiss, Virotte-Ducharme, and Zara.)

For several years Aschbacher has worked on a program for organizing the theory of the sporadic finite simple groups. There are three basic parts to his program for each group — existence, uniqueness (given a suitable involution centralizer), and local structure (as revealed in the normalizers of all subgroups of prime order). In his previous book [5] he presented basic material in aid of this program and carried through the steps for five of the sporadic groups. The second and third parts of the present book do this for the three sporadic Fischer groups. As such, these parts are more demanding and technical than the first and are aimed at a more specialized audience. Even so, almost all the background comes from Aschbacher's earlier books [4], [5], and the most important matters are presented in preliminary chapters to each part. (The sole exception is appeal to Glauberman's $Z^*$-theorem [14].)

Part II is devoted to existence and uniqueness for these groups. Uniqueness comes from the characterizations of Part I. Aschbacher proves existence by locating the groups as sections of the sporadic simple group known as the Monster, having already constructed that group in [5]. (A section of a group is a homomorphic image of a subgroup.) The argument and interaction are delicate, since the hypotheses of Fischer's characterization are not readily verified within the Monster. In any case, the global 3-transposition hypothesis is not a suitable local hypothesis for uniqueness questions. Aschbacher deals with this by examining a number of related situations, each with its own virtues and disadvantages, and then proving that they coexist.

Following Aschbacher, we say that a group has type $M(22)$ if it is a 3-transposition group with trivial center and local type $\mathcal{F}(U_6(2), L)$ (as above). It turns out that there is only one possibility for $L$, which is essentially a unitary group in dimension 4 over $GF(9)$ and is denoted $Z_2/U_4(3)/Z_3$. The specification of $L$ becomes part of Aschbacher's definition of type $M(22)$ as well. Next a group has type $M(23)$ if it is a 3-transposition group with trivial center and local type $\mathcal{F}(M(22), L)$ (for the appropriate and uniquely determined $L$), and a group with trivial center has type $M(24)$ if it is a 3-transposition group of local type $\mathcal{F}(M(23), L)$, again for appropriate and unique $L$. Finally, a group has type $M(24)'$ if it is the derived subgroup of a group of type $M(24)$. The classification theorem of Part I guarantees uniqueness for any group of type $M(22), M(23), M(24)$ or $M(24)'$, but it is silent as to whether such groups exist.

Aschbacher's existence proof for the Fischer groups begins with a section of the Monster whose structure resembles that of $M(24)$. Specifically, it has a simple subgroup of index 2 containing an involution $z$ whose centralizer $C(z)$ has three properties:

1. $C(z)$ has a normal extraspecial subgroup $Q$ of order $2^{13}$ which contains its own centralizer in $C(z)$,
2. $C(z)/Q$ is a group $Z_2/U_4(3)/Z_3$,
3. $Q$ contains conjugates of $z$ other than $z$ itself.

Aschbacher says that any finite group satisfying these three properties has type $F_{24}$. A group that has an index 2 subgroup of type $F_{24}$ (and also satisfies an appropriate non-degeneracy condition) is said to have type $Aut(F_{24})$. In view of the Monster section, existence is established for such groups, but uniqueness is unclear. Groups of type $\tilde{M}(22)$, $\tilde{M}(23)$, and $\tilde{M}(24)$ are each defined to have an involution whose centralizer models that of a 3-transposition in the corresponding group, but no global assumption on the class is made.

In the successive Sections 31, 32, and 33, we find that type $\tilde{M}(n)$ implies type $M(n)$ for $n$ equal to 22, 23, 24, respectively. In particular, if groups of type $\tilde{M}$ exist, then they are unique since they come from the groups $M$ already known to be unique by Part I. The group $M(24)$ is encountered again in Section 35, where it is shown that a group of type $Aut(F_{24})$ also has type $M(24)$. In particular, thanks to the Monster, a group of type $M(24)$ does exist, and therefore so do those of type $M(22)$ and $M(23)$. Furthermore, groups of type $Aut(F_{24})$ are unique, since they must come from groups of type $M(24)$, already unique. Finally, Sections 34 and 36 contain the proof that a group of type $F_{24}$ must have type $M(24)'$. Thus the latter group exists and the former group is unique. And they are the same! The interaction of the uniqueness and existence arguments is delicate, and Aschbacher must take great care to avoid any suggestion of circular reasoning.

Most of the arguments are typical of $p$-local analysis. Additionally, the simplicial methods of Aschbacher and Segev [5, Chap. 12] are used to measure simple connectivity for the commuting graphs of several 3-transposition groups. These results are then employed within the uniqueness arguments to construct subgroups $M(23)$ and $M(24)'$ within any group of type $F_{24}$. Some cohomological results are also presented in order to construct the central and module extensions needed.

Part III is the shortest of the three parts and holds the fewest surprises. For each of the Fischer groups, the normalizers of all subgroups of prime order are listed. These are found using standard local techniques. The arguments actually treat the full automorphism group of each Fischer group. Here $M(24)$ is the automorphism group of $M(24)'$, and $M(23)$ is its own automorphism group, but $M(22)$ has index 2 in its automorphism group.

The classification of finite simple groups careered through the 1970's with a torrent of lengthy and difficult papers, Aschbacher himself prominent among the authors. The pace and magnitude of this exhilarating project produced, as byproduct, disincentive for extended reflection, motivation, insight, detail, or exposition. In modern jargon, the subject was more "goal-oriented" than "user-friendly". Aschbacher's revised treatment of the sporadic groups, as presented (so far) in this and his previous book [5], responds to several of the earlier omissions. Many of the arguments are still difficult and dry, and there are places where the reasoning is more efficient than transparent. But Aschbacher keeps his reader firmly in mind and has taken pains to put all his cards on the table. Each part has a good introduction, which provides location, motivation, and summary. With only a very few exceptions (such as the $Z^*$-theorem) all the material needed to understand Aschbacher's book is contained there or in one of the two previous books [4], [5]. He prefers to provide needed background, even in situations where it would be easy to refer elsewhere. Such is the case with the cohomological and simplicial methods of Part II, reviewed in the present book and covered in detail in [5]. The various background sections are more than standard supplements but instead are given from Aschbacher's unique and insightful point of view. Thus Parts II and III contain chapters which stand on their own and are worthwhile reading even for those without specific interest in the uniqueness, existence, and local results of those two parts. We mention two cases as illustration. In Chapter 10 of Part II he presents and analyzes a lattice over $\mathbf{Z}[e^{2\pi i/3}]$ in order to explain and elucidate the embedding of the dimension 4 unitary group over $GF(9)$ within the dimension 6 unitary group over $GF(4)$. This approach is very similar to the treatment that Conway and others have given to the Leech lattice and related lattices (see [7]). In Chapter 13 of Part III, he presents a complete parameterization of all classes of elements of order 3 in orthogonal groups over $GF(3)$ in an approach similar to the classic work of Wall [19].

In summary, Aschbacher's excellent book is required for all with specific interest in finite simple groups, but those with a less direct connection will also find much of value. In particular, they will find the only available collected treatment of Fischer's classification of 3-transposition groups, one of the most important and historic results from the theory of finite simple groups, presented lucidly by one of the most original minds in that area.

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