BOOK REVIEWS

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Some points of analysis and their history, by Lars Gårding, University Lecture Series, vol. 11, Amer. Math. Soc., Providence, RI, 1997, 88+vii pp., $16.00, ISBN 0-8218-0757-9

This lively book is a guided tour through some of the highlights of twentieth-century analysis. It has the strengths and weaknesses of a series of lectures: it is personal, readable, uneven, and occasionally repetitive rather than anonymous and organized to perfection. All three of the standard subunits of modern analysis – complex, real, and functional – are represented, the latter two especially in the context of PDEs.

Three of the first four chapters form a unit that deals with the growth properties of analytic functions, from Picard's great theorem through Phragmén-Lindelöf to the Nevanlinna theory and the early work of Ahlfors. Most of the important ideas and proofs are touched upon, and many are sketched concisely but informatively. The essentials of the Möbius group and its basic properties, tesselations of the upper half plane, automorphic functions, and Picard's proof that a function with essential singularity at a point can omit at most one value are described in three pages, and ideas of Borel and Schottky linking the theorem to growth properties take up two more. The chapter on the Phragmén-Lindelöf principle – that a function holomorphic in a sector and bounded on the boundary either grows quite fast or attains its maximum modulus on the boundary – locates the roots of Phragmén's first paper in the work of Mittag-Leffler on generalizations of the exponential function. The main ideas of papers of Phragmén, of Phragmén and Lindelöf, and of Carleman are sketched, as well as a number of applications of the principle. The chapter on Nevanlinna theory begins with Jensen's formula for a function analytic in the disc $\{|z| \le r\}$ with zeros only at $a_1, \dots a_n \neq 0$:

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta = \log|f(0)| + \log\frac{r^n}{|a_1\cdots a_n|} \tag{1}$$

and explores the implications through outlines of Nevanlinna's First and Second Fundamental Theorems in four concentrated pages. The culmination is a condensed discussion of the geometric and topological approaches to the Second Fundamental Theorem that were introduced by Ahlfors.

Complex analysis is also the background for the chapter on the Riesz-Thorin interpolation theory, though ultimately the implications for real analysis have been

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more profound. The starting point is the Hilbert transform

$$Tu(x) = \frac{1}{\pi}\text{p.v.}\int_{-\infty}^{\infty} \frac{u(t)\,dt}{x-t} \tag{2}$$

that relates the real and imaginary parts, on the line, of a function holomorphic and small at infinity in the upper half plane. Hilbert proved that T is an isometry in $L^2(\mathbb{R})$. Gårding outlines the two proofs by Marcel Riesz that T is bounded in L^p, $1 < p < \infty$, and the proof of Thorin of Riesz's convexity result, then sketches several applications.

The general emphasis on the work of Swedish and Finnish analysts is interrupted by the two middle chapters. The first gives a thumbnail sketch of abstract harmonic analysis after an efficient presentation of the properties of the Fourier transform on $L^1(\mathbb{R})$ and a modern adaptation of Wiener's proof of the Wiener Tauberian Theorem: the translates of $f \in L^1$ are dense in L^1 if and only if the Fourier transform $\hat{f}$ has no zeros. The Gel'fand version for the case of Fourier series is followed by a survey of subsequent developments in harmonic analysis on locally compact abelian groups and Beurling's sharpening of Wiener's theorem. The chapter ends with the question of spectral synthesis: the relation between general ideals in the convolution algebra and ideals determined by closed subsets of the maximal ideal space.

The second middle chapter presents a "leisurely" exposition of Hörmander's version of Paul Cohen's proof of a theorem of Tarski known mainly through the version by Seidenberg, which, on its face, has little to do with analysis: if $A \subset \mathbb{R}^n$ consists of all points that satisfy some subset of a finite set of polynomial inequalities, then any projection of A to a linear subspace is a set of the same type. A corollary: let p be a polynomial on $\mathbb{C}^n$. If, given that $p(z) = 0$, $|z| \to \infty$ implies that $|\operatorname{Im} z| \to \infty$, then $|\operatorname{Im} z|$ grows at an algebraic rate with respect to $|z|$.

The author is an originator of the modern approach to PDE through systematic use of the Fourier transform and functional analysis. An early chapter covers uniqueness theorems for the Cauchy problem. The (linear) Cauchy-Kovalevskaya theorem says that a system

$$\frac{\partial u}{\partial x_1} + \sum_{j=2}^{n} A_j(x)\frac{\partial u}{\partial x_j} = f, \qquad u|_{x_1=0} = g, \tag{3}$$

with coefficients A_j and data f, g analytic near 0 has a unique solution u analytic near 0. Gårding gives Holmgren's duality argument proving that there is a unique C^1 solution and notes that uniqueness fails in general if the coefficients are merely smooth or if the equation is nonlinear. He then outlines a generalization of Carleman's uniqueness theorem for non-analytic coefficients and mentions further fundamental results of Hörmander and of Calderón.

In the chapters that follow the Tarsky-Seidenberg excursion, the topic is PDE. Two of the author's own innovations lead them off. The distinction among types of second order equations—elliptic, parabolic, and hyperbolic—and the fact that the naturally associated boundary value problems are quite different had been long known. Gårding may be the first to formulate and prove an exact relation between an algebraic and an analytic property for a general constant coefficient differential operator

$$P(D), \quad D_j = \frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_j}, \qquad p(\xi) = \sum_{|\alpha| \le m} a_\alpha\, \xi^\alpha : \tag{4}$$

P is intrinsically hyperbolic with respect to the x_1 direction if and only if the zeros τ of $p(\tau, \xi_2, \ldots, \xi_n)$ have bounded imaginary part as $(\xi_2, \ldots, \xi_n)$ ranges through $\mathbb{R}^{n-1}$. Here "intrinsic hyperbolicity" means that solutions of $Pu = 0$ tend to zero uniformly on bounded sets if $D_1^j u|_{x_1=0}$ tend to zero uniformly on bounded sets, $0 \leq j < m$. Gårding deduces this here from Tarski-Seidenberg. (Originally that theorem was not available, and he proved the necessary lemma directly). Tarski-Seidenberg is used in the same way in the chapter on Hörmander's characterization of hypoelliptic operators $P = p(D)$: all (distribution) solutions u of $Pu = 0$ in a non-empty open set Ω are smooth in Ω if and only if $p(z) = 0$, $|z| \to \infty$ implies $|\operatorname{Im} z| \to \infty$. Here the bridge between necessity and sufficiency is supplied by the corollary mentioned above and a strengthened form.

The Dirichlet problem – find a function harmonic on a bounded plane domain that has prescribed boundary values – has played a central role in nineteenth- and twentieth-century analysis. Gårding sketches this role briefly, leading to Weyl's approach. Weyl separates the questions of regularity and of existence and deduces existence of a weak solution from the inequality

$$-\operatorname{Re}\Big(\int_\Omega \bar{u}(x)\,\Delta u(x)\,dx\Big) \geq 0 \quad \text{if} \quad u = 0 \ \text{ on } \ \partial\Omega, \tag{5}$$

which follows immediately from an integration by parts. The generalization to second order operators with variable coefficients is immediate. Gårding generalized to higher (even) order operators by proving Gårding's inequality:

$$\operatorname{Re}\Big(\int_\Omega \bar{u}\,Pu\,dx\Big) \geq (c-\varepsilon)\sum_{|\alpha|\leq 2m} ||D^\alpha u||^2 - b_\varepsilon ||u||^2, \quad u \equiv 0 \ \text{ near } \ \partial\Omega. \tag{6}$$

Here P is an operator with smooth coefficients whose principal part satisfies

$$\operatorname{Re}(p_{2m}(x,\xi)) = \operatorname{Re}\Big(\sum_{|\alpha|=2m} a_\alpha(x)\,\xi^\alpha\Big) \geq c \sum_{|\alpha|=2m} |\xi^\alpha|^2, \qquad \xi \in \mathbb{R}^n; \tag{7}$$

$||\ ||$ denotes the L^2 norm.

Gårding's inequality has a "sharp" form due to Hörmander in which one only assumes $\operatorname{Re} p(x,\xi) \geq 0$, and P can be a pseudodifferential operator. The penultimate, and least successful, chapter is devoted to this sharp form. It opens with "a short and self-contained description of the calculus of pseudodifferential operators." A reader who is already comfortable with the basics, or is willing to take quite a lot of background and motivation on faith, can then make it through Gårding's presentation of a more recent proof by Hörmander, but there is nothing to suggest that the result is of any but purely technical interest.

The final chapter, "The impact of distributions in analysis", is a brief account of the reception of Laurent Schwartz's theory and its ultimate incorporation into every analyst's tool kit. Here the author sketches one important proof: of the theorem of Ehrenpreis and Malgrange that every constant coefficient partial differential operator P has a fundamental solution, so that $Pu = f$ has a solution whenever f has compact support.

The book covers a wide range of material, much of which should be of interest to any student of analysis. It allows one to see much of the terrain from the point of view of a major participant as the threads of hard and soft analysis were woven together in the latter part of the century. This is clearly not a textbook, but it is far from a history – one learns a bit about who taught at which Swedish university, who was whose student, and what might have motivated some of the mathematics, but after a promise to explain the origin of the “unnecessarily complicated name” of Wiener’s Tauberian theorem comes a paragraph that mentions asymptotics but neither Abel nor Tauber. However, there are interesting tidbits, like the famous thesis which, even by the time of the public defense, had not been read by the advisor.

For use, rather than perusal, the book can be recommended, for example, to a graduate student seminar. By filling in the details and doing a bit more reading in the references as necessary, one can pick up a number of the gems of analysis without having to mine whole seams. But let the user beware: the informal lecture format, when not carefully edited, leaves in some clinkers. Some are merely annoying typos, like the equality in the middle of page 39 that should be an inequality, or minor careless points, like the introduction without explanation of TL^1 (later replaced by $\mathcal{F}L^1$) for the image under the Fourier transform. After stating that Laurent Schwartz proved that the answer to the question at the top of page 46 is “no”, Gårding sketches Schwartz’s proof that the answer is “yes”. The usual quick proof of Malgrange’s lemma has been replaced at the top of page 81 by an argument which would imply that $|1 - e^{i\theta}|^{-1}$ is integrable on $[0, 2\pi]$. And Gårding’s inequality, (4) on page 68 and (6) above, does not follow (with the same constant c) from Gårding’s assumption (3); one needs the form (7) above.

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