RESEARCH REPORTS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 35, Number 2, April 1998, Pages 123–143
S 0273-0979(98)00745-9

VOEVODSKY'S PROOF OF MILNOR'S CONJECTURE

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Abstract. We give an overview of Voevodsky's recent proof of Milnor's conjecture on the mod 2 Galois cohomology of fields of characteristic $\neq 2$.

1. Introduction

In [43], V. Voevodsky gives a proof of Milnor's conjecture on the Galois cohomology of fields of characteristic $\neq 2$ [24].

The proof starts from his joint work with A. Suslin [40] on *motivic cohomology*, which allows Voevodsky to reformulate Milnor's conjecture as higher dimensional analogues of Hilbert's Theorem 90. Voevodsky's insight is that these analogues are inductively implied by the vanishing, for each $n \geq 0$, each field k (of characteristic zero), each n-uple $\underline{a} = (a_1, .., a_n) \in (k^*)^n$, of some motivic cohomology group of a suitable *simplicial smooth k-variety* associated to $\underline{a}$. Voevodsky reformulates two results of M. Rost [30], [31] on the Chow motives of quadrics as the vanishing of a higher dimensional motivic cohomology group. Miraculously, the group he wants to vanish maps to the one he knows to vanish using an explicit *cohomology operation*. Voevodsky then proves such strong properties of the action of these operations on the motivic cohomology of these simplicial smooth k-varieties that the comparison homomorphism has to be injective, proving the required vanishing.

In these notes, I will try to give the reader some ideas both of the proof and of the power of these new techniques, successfully brought from algebraic topology to algebraic geometry by Voevodsky. For more details, the reader may also consult [13], [10] and of course [43].

Notations. Everywhere in the text, k denotes a field and k_s a separable closure of k. For any Galois extension $F \subset L$, $G(L|F)$ denotes the Galois group of this extension.

Received by the editors October 7, 1997, and in revised form January 25, 1998.
1991 *Mathematics Subject Classification.* Primary 12G05, 14C25, 55P42, 55S10, 57R20.
Key words and phrases. Galois cohomology, algebraic cycles, motives, stable homotopy theory.

2. Galois cohomology and Milnor K-theory

In this section we assume $char(k) \neq 2$. Let $n \geq 0$ be an integer and let $H^n(G(k_s|k);\mathbf{Z}/2)$ be the n-th cohomology group of the profinite group $G(k_s|k)$ with coefficients in $\mathbf{Z}/2$ (*cf.* [35] for the basic definitions). These groups altogether form a graded, connected, commutative $\mathbf{Z}/2$-algebra which we denote $H^*(G(k_s|k);\mathbf{Z}/2)$.

As $H^1(G(k_s|k);\mathbf{Z}/2)$ is the group of continuous group homomorphisms from $G(k_s|k)$ to $\mathbf{Z}/2$, the elementary theory of quadratic extensions defines an isomorphism $h_1(k) : k^*/2 \cong H^1(G(k_s|k);\mathbf{Z}/2)$. But for our purpose it is convenient to define this isomorphism as follows. The Kummer exact sequence of discrete $G(k_s|k)$-modules

$$0 \to \{\pm 1\} \to k_s^* \stackrel{(-)^2}{\to} k_s^* \to 0$$

(where $\{\pm 1\}$ is the group of square roots of unity in k with trivial action of the Galois group) determines a long exact sequence of cohomology groups (using the canonical isomorphism of groups between $\{\pm 1\}$ and $\mathbf{Z}/2$) :

$$0 \to \{\pm 1\} \to k^* \stackrel{(-)^2}{\to} k^* \stackrel{\delta(k)}{\to} H^1(G(k_s|k);\mathbf{Z}/2) \to H^1(G(k_s|k), k_s^*) \to ...$$

Hilbert's Theorem 90, which essentially says that $H^1(G(k_s|k);k_s^*)$ vanishes [34], implies that $\delta(k)$ induces our isomorphism :

$$h_1(k) : k^*/2 \cong H^1(G(k_s|k);\mathbf{Z}/2).$$

For any element a in k^*, we shall simply denote $(a) \in H^1(G(k_s|k);\mathbf{Z}/2)$ its image by $\delta(k)$.

Lemma 2.1 (Bass, Tate [24])**.** *For any $a \in k^* - \{1\}$, the product $(a).(1-a)$ is zero in $H^2(G(k_s|k);\mathbf{Z}/2)$.*[1]

The *Milnor K-theory* of k [24] is the graded associative ring with unit, denoted $K_*^M(k)$, quotient of the tensor algebra over the group k^*, placed in degree one, by the two-sided ideal generated by elements of the form $a \otimes (1-a)$ for any $a \in k^* - \{1\}$. This graded associative ring is in fact commutative (in the graded sense). We obviously have $K_0^M(k) = \mathbf{Z}$ and $K_1^M(k) = k^*$.

Lemma 2.1 above implies that the homomorphism $\delta(k) : k^* \to H^1(G(k_s|k);\mathbf{Z}/2)$, $a \mapsto (a)$ induces a homomorphism of graded commutative $\mathbf{Z}/2$-algebras :

$$h_*(k) : K_*^M(k)/2 \to H^*(G(k_s|k);\mathbf{Z}/2)$$

(sometimes called the *norm residue homomorphism* or the *Galois symbol*). It is an isomorphism in degrees ≤ 1. The main result of [43] is :

Theorem 2.2. [43] *For any field k of characteristic different from 2, the homomorphism*

$$h_*(k) : K_*^M(k)/2 \to H^*(G(k_s|k);\mathbf{Z}/2)$$

is an isomorphism.

[1] If a is not a square in k, let $L \subset k_s$ be the quadratic extension of k generated by the square roots of a. The lemma is proven using the *projection formula* involving the *restriction* homomorphism $Res_{L|k} : H^*(G(k_s|k);\mathbf{Z}/2) \to H^*(G(k_s|L);\mathbf{Z}/2)$, and the *corestriction* homomorphism $Cor_{L|k} : H^*(G(k_s|L);\mathbf{Z}/2) \to H^*(G(k_s|k);\mathbf{Z}/2)$ [35].

This result was conjectured by J. Milnor [24] and proved there for global fields (using computations of $K_2^M(k)/2$ for global fields by H. Bass and J. Tate [2]) as well as for finite, local and real closed fields. A. Merkurjev [19] proved the degree 2 case in general, M. Rost [29] and independently A. Merkurjev and A. Suslin [21] the degree 3 case. Some years ago, the proof of the degree 4 case was announced by M. Rost.

Remark 2.3. The reader should notice that, among others, an amazing consequence of this theorem is that the graded ring $H^*(G(k_s|k);\mathbf{Z}/2)$ is generated as a ring by elements of degree 1.

Milnor's conjecture has been generalized to an arbitrary prime $\ell \neq char(k)$ by Kato [14]; see 4.5 below. (Bloch considered a weaker version in [4].) Most of Voevodsky's arguments work for any prime ℓ. The reason he has to assume $\ell = 2$ at the moment is the lack of objects which should play the role that quadrics and quadratic forms play at $\ell = 2$.

In order to attack the proof of Milnor's conjecture we shall need the basic definitions and properties of motivic cohomology of smooth varieties over k given in [40], [45] and [11]. This is done in the next section. In section 4 below, Milnor's conjecture is seen to be equivalent to higher dimensional analogues of Hilbert's theorem 90, thus generalizing the degree 1 case. In section 5 the general strategy of the proof of these analogues is given. The next two sections, 6 and 7, detail the final steps.

3. Motivic cohomology

In the appendix of [3], A. Beilinson conjectured the existence of a *motivic cohomology* theory for schemes together with basic properties, which are partly established for the motivic cohomology of smooth k-varieties defined by Suslin-Voevodsky. Motivic cohomology aims to play for algebraic varieties the role played for CW-complexes by singular cohomology.

Let us recall a way to define singular cohomology with integral coefficients. The Eilenberg-Mac Lane (topological) space generated by a CW-complex Y is the free topological abelian group generated by Y. It is the group completion $L^{top}[Y]$ of the topological commutative monoid $\amalg_{n\geq 0} SP^n(Y)$, where for each $n \geq 0$, $SP^n(Y)$ denotes the n-th symmetric product of Y. From the Dold-Thom theorem [8] its homotopy groups are the singular homology groups of Y with $\mathbf{Z}$ coefficients. Let $n \geq 0$, S^n be the n dimensional sphere (homeomorphic to the smash product $S^1 \wedge .. \wedge S^1$ of n pointed circles) and L_n^{top} be the quotient of $L^{top}[S^n]$ by the relation : base point $= 0$. Again, the Dold-Thom theorem implies that the m-th homotopy group of L_n^{top} is trivial except when $m = n$, in which case it equals $\mathbf{Z}$.

Recall that for each $n \geq 0$ the standard topological n-simplex, Δ^n_{top}, is the subspace of $\mathbf{R}^{n+1}$ defined by the equations $\Sigma_{i\in\{0,\ldots,n\}} x_i = 1$ and $x_i \geq 0$ for each i. Together with the *cofaces* $d^i : \Delta^{n-1}_{top} \to \Delta^n_{top}$ which send $(x_0,\ldots,x_{n-1})$ to $(x_0,\ldots,x_{i-1},0,x_i,\ldots,x_{n-1})$ and the *codegeneracies* $s^i : \Delta^n_{top} \to \Delta^{n-1}_{top}$ which send $(x_0,\ldots,x_n)$ to $(x_0,\ldots,x_i+x_{i+1},\ldots,x_n)$, they form a *cosimplicial* topological space, denoted $\Delta^\bullet_{top}$.

For any CW-complex X, one can form the complex of abelian groups associated to the simplicial abelian group $i \mapsto Hom_{top}(X\times\Delta^i_{top}, L_n^{top})$ ($Hom_{top}(-,-)$ meaning continuous maps). It is well known that the m-th homology group of this complex is naturally isomorphic to the $(n-m)$-th singular cohomology group of X with $\mathbf{Z}$-coefficients, for $m \leq n$

Following Suslin's definition of singular homology of algebraic varieties [41], [17], the definitions of motivic cohomology and motivic chain complexes of smooth k-varieties are largely inspired by this picture.

3.1. **Eilenberg-Mac Lane objects generated by smooth varieties.** Let $Sm(k)$ denote the category of smooth quasi-projective varieties over k, called smooth k-varieties in the sequel. For X and Y in $Sm(k)$, let $c(X,Y)$ be the free abelian group on the set of closed irreducible subvarieties Z of $X \times_k Y$ for which the projection $Z \to X$ is *finite and surjective onto an irreducible component of* X (for example, the graph of a morphism to Y from an irreducible component of X). Elements of $c(X,Y)$ are called *finite correspondences* from X to Y.

Let X_1, X_2 and X_3 be smooth k-varieties. Using standard intersection theory (for example [36]), one constructs natural homomorphisms of groups : $c(X_1,X_2) \otimes c(X_2,X_3) \to c(X_1,X_3)$ compatible with the composition of morphisms. In this way we get a category $SmCor(k)$ as follows : its objects are smooth k-varieties, the set of morphisms from X to Y is the set $c(X,Y)$ of finite correspondences from X to Y, and the composition is defined using the above homomorphisms. This category has finite sums and is additive.

For any smooth k-variety X, let us denote $L[X]$ the functor :

$$(SmCor(k))^{op} \to Ab,$$

$$Y \mapsto c(Y,X)$$

and call it the *Eilenberg-Mac Lane object generated by* X. This terminology is suggested by the following result.

Let V_k be the category of quasi-projective k-varieties (k-varieties in the sequel). For any smooth k-variety Y and any $n \geq 0$, one constructs the n-th symmetric product of Y, denoted $SP^n(Y) \in V_k$, as the quotient of Y^n by the action of the n-th symmetric group (*cf.* [33] for instance). It is singular if and only if Y has an irreducible component of dimension ≥ 2 and $n \geq 2$.

In [41] Suslin and Voevodsky construct for X and Y smooth k-varieties a natural isomorphism of groups :

$$c(X,Y)[1/p] \cong (Hom_{V_k}(X, \amalg_{n\geq 0} SP^n(Y)))^+[1/p],$$

where the $^+$ symbol means the group completion of a commutative monoid and $[1/p]$ means that we invert p, the characteristic exponent of k, *i.e.* $p = char(k)$ if $char(k) \neq 0$ and $p = 1$ if $char(k) = 0$.

3.2. **Motivic chain complexes.** Let $n \geq 0$ and let Δ^n be the closed subscheme in the affine $(n+1)$-space $\mathbf{A}_k^{n+1}$ defined by the equation $\Sigma_{i=0,\dots,n} x_i = 0$. Of course, it is isomorphic to $\mathbf{A}_k^n$, but the above presentation allows one to define a *cosimplicial* object $\Delta^\bullet$ in $Sm(k)$ from the collection of Δ^n, $n \geq 0$, using the same formulas (for the d^i's and s^i's) as in the topological case.

Let us say that a *presheaf of abelian groups* (on $Sm(k)$) is any functor F : $(Sm(k))^{op} \to$ abelian groups and that a *presheaf of complexes* (on $Sm(k)$) is any functor $(Sm(k))^{op} \to$ chain complexes of abelian groups.

For any presheaf of abelian groups F on $Sm(k)$ one defines its *singular chain complex* $\underline{C}_*(F)$ as follows. It is the presheaf of complexes which sends a smooth k-variety X to the chain complex $\underline{C}_*(F)(X)$ associated to the simplicial abelian group $n \mapsto F(X \times \Delta^n)$ whose differential in homological degree $n \geq 0$ is given by the formula $\Sigma_{i=0,..,n}(-1)^i F(d^i)$. For any smooth k-variety X, we simply denote $\underline{C}_*(X)$ the presheaf of complexes $\underline{C}_* L[X]$ and call it the *motivic chain complex of* X. (Following A. Suslin, the *singular homology groups* $H^S_*(X)$ of X are the homology groups of the chain complex $\underline{C}_*(X)(Spec(k))$ [41] and also [17].)

Let $\mathbf{G}_m \subset \mathbf{A}^1$ be the complement of $0 \in \mathbf{A}^1$. For any $i \geq 0$, let L_i be the cokernel (in the category of presheaves of abelian groups) of the morphism :

$$\oplus_{j=1,..,i} L[(\mathbf{G}_m)^{i-1}] \to L[(\mathbf{G}_m)^i]$$

given by the sum of the i morphisms induced by the obvious inclusions $(\mathbf{G}_m)^{i-1} \subset (\mathbf{G}_m)^i$ which set the j-th coordinate equal to 1.

In view of the analogy above, L_i should be considered as the algebraic analogue of the topological abelian group L_i^{top} above : indeed, the latter is the quotient (in the category of topological abelian groups) of $L^{top}[S^1 \times .. \times S^1]$ by the sum of the image of the corresponding inclusions.

Finally, let $\mathbf{Z}(i)$ be the presheaf of complexes (on $Sm(k)$) equal to $\underline{C}_*(L_i)[-i]$, the i-th desuspension of $\underline{C}_*(L_i)$. (The suspension sends things of homological degree n to homological degree $n+1$ and the desuspension does the reverse.)

Definition 3.1. Let X be a smooth k-variety and $i \geq 0$. The Zariski hypercohomology of X with coefficients in $\mathbf{Z}(i)$ is denoted $H^*_B(X; \mathbf{Z}(i))$. The bigraded abelian group $H^*_B(X; \mathbf{Z}(*))$ is called the *Beilinson motivic cohomology* (or, in short, the *motivic cohomology*) of X with coefficients in $\mathbf{Z}$.

Remark 3.2. For any abelian group A, and any smooth k-variety X, one defines the motivic cohomology of X with coefficients in A, denoted $H^*_B(X; A(*))$, using the presheaf of complexes $\mathbf{Z}(i) \otimes A$ whose value on any smooth k-variety is the naive tensor product. Any short exact sequence of abelian groups yields a long exact sequence of the corresponding motivic cohomology groups with coefficients.

3.3. **Basic properties of motivic cohomology.** Observe first that the motivic cohomology of a smooth k-variety X only depends on the underlying scheme (because so does the restriction of the complexes $\mathbf{Z}(i)$ to X).

Multiplicative structure. For any commutative ring with unit A and any smooth k-variety X the bigraded abelian group $H^*_B(X; A(*))$ has a natural structure of bigraded commutative associative ring with unit. The product is denoted by $\smile$ and the bigraded commutativity of $H^*_B(X; A(*))$ means that for any $\alpha \in H^n_B(X; A(i))$ and $\beta \in H^m_B(X; A(j))$, one has : $\beta \smile \alpha = (-1)^{nm} \beta \smile \alpha$ in $H^{n+m}_B(X; A(i+j))$.

Vanishing. By definition, each presheaf of complexes $\mathbf{Z}(i)$ vanishes in homological dimensions $\leq -i-1$. It follows that the motivic cohomology group $H^n_B(X; A(i))$ vanishes if $n > i + d_X$, where d_X is the Krull dimension of X. For example, $H^n_B(Spec(k); A(i)) = 0$ if $n > i$. It is conjectured (Beilinson and Soulé's *vanishing conjecture*) that for any smooth k-variety X the motivic cohomology group $H^n_B(X; \mathbf{Z}(i))$ vanishes if $n < 0$.

Low weights. For any smooth k-variety X we obviously have :

$$H^n_B(X; \mathbf{Z}(0)) = \begin{cases} 0 \text{ if } n \neq 0 ; \\ \mathbf{Z}^{\pi_0(X)} \text{ for } n = 0, \end{cases}$$

where $\pi_0(X)$ is the set of connected components of X and it can be shown using results from [41] that there are natural isomorphisms :

$$H^n_B(X;\mathbf{Z}(1)) \cong \begin{cases} 0 \text{ if } n \neq 1,2 \ ; \\ \mathcal{O}(X)^* \text{ for } n=1 \ ; \\ Pic(X) \text{ for } n=2 \ . \end{cases}$$

Milnor's K-theory. It follows that we have a canonical isomorphism :

$$k^* \cong H^1_B(Spec(k);\mathbf{Z}(1)),$$
$$a \mapsto (a).$$

Let $H^*_B(k)$ be the graded commutative ring equal in degree n to $H^n_B(Spec(k);\mathbf{Z}(n))$.

Proposition 3.3. [40] *For any field k, any $a \in k^* - \{1\}$, one has $(a) \smile (1-a) = 0$ in $H^2_B(Spec(k);\mathbf{Z}(2))$ and the induced homomorphism of graded rings*

$$K^M_*(k) \to H^*_B(k)$$

is an isomorphism.

The proof of this proposition relies among others things on the existence of suitable transfers in Milnor K-theory [2], [14].

Corollary 3.4. *For any field k, any integers $n \geq 0$ and $m > 0$ the induced homomorphism of abelian groups :*

$$K^M_n(k)/m \to H^n_B(Spec(k);\mathbf{Z}/m(n))$$

is an isomorphism.

This follows from the long exact sequence associated to the exact sequence $0 \to \mathbf{Z}(n) \stackrel{\cdot m}{\to} \mathbf{Z}(n) \to \mathbf{Z}/m(n) \to 0$ and the fact that $H^{n+1}_B(Spec(k);\mathbf{Z}(n)) = 0$.

4. Étale motivic cohomology

4.1. **Changing the topology.** For any $i \geq 0$ and any abelian group A, the presheaf of complexes $A(i)$ can be used to produce hypercohomology groups in any Grothendieck topology on $Sm(k)$. In this section we study hypercohomology of the $A(i)$'s in the *étale* topology. (See [48], [22].)

Definition 4.1. For any $i \geq 0$, any abelian group A and any smooth k-variety X we denote :

$$H^*_L(X;A(i))$$

the hypercohomology groups of X in the *étale topology* of the presheaves of complexes $A(i)$, and call them the *Lichtenbaum motivic cohomology groups* (or *étale motivic cohomology groups*) of X with coefficients in A.

S. Lichtenbaum developed several conjectures concerning the existence and properties of these étale motivic cohomology groups [18]. We have a canonical comparison homomorphism of bigraded abelian groups :

$$H^*_B(X;A(*)) \to H^*_L(X;A(*)).$$

Lemma 4.2. [44] *For any field k, any smooth k-variety X, any $\mathbf{Q}$-vector space A, the homomorphism*

$$H^*_B(X;A(*)) \to H^*_L(X;A(*))$$

is an isomorphism.

If the field k admits the resolution of singularities, it is shown in [45] (using [11], [39]) that motivic cohomology of equidimensional smooth k-varieties agrees with Bloch's higher Chow groups [5]. Thus from [6], [16], it follows that for such smooth k-varieties, motivic cohomology with rational coefficients coincides with that defined by Beilinson using K-groups [3].

Let A be a torsion abelian group of torsion prime to $char(k)$. Let $A_{ét}$ be the constant étale sheaf associated to A [48], and for any $i \geq 0$ let $A_{ét}(i)$ be its i-th *Tate twist* [48]. For example, for n prime to $char(k)$, $(\mathbf{Z}/n)_{ét}(1)$ is the étale sheaf of n-th roots of unity μ_n, and $(\mathbf{Z}/n)_{ét}(i)$ is its i-th tensor power $\mu_n^{\otimes i}$. As far as torsion coefficients (of torsion prime to $char(k)$) are concerned, étale motivic cohomology groups are just the corresponding étale cohomology groups defined and studied by A. Grothendieck and others [48] :

Lemma 4.3. [40] *Let A be a* torsion abelian group of torsion prime to $char(k)$. *Then for any $i \geq 0$ and any smooth k-variety X, there are canonical isomorphisms :*

$$H^n_L(X; A(i)) \cong H^n_{ét}(X; A_{ét}(i)).$$

In fact the complex of étale sheaves $A(i)_{ét}$ associated to $A(i)$ is shown to be quasi-isomorphic to the étale sheaf $A_{ét}(i)$ (placed in degree 0). This is more or less a direct consequence of the fact the sequence of presheaves $0 \to \mu_n \to \mathbf{G}_m \overset{(-)^n}{\to} \mathbf{G}_m \to 0$ becomes exact in the category of étale sheaves of abelian groups.

4.2. **Higher dimensional analogues of Hilbert's theorem** 90**.** Let ℓ be a prime number $\neq char(k)$. As in the case $\ell = 2$, for each $n \geq 0$, one may define canonical homomorphisms :

$$h_n(k) : K^M_n(k)/\ell \to H^n(G(k_s|k); \mu_\ell^{\otimes n}).$$

The discrete $G(k_s|k)$-module $\mu_\ell^{\otimes n}$ is the n-th tensor power of the group $\mu_\ell(Spec(k_s))$ of ℓ-th roots of unity in k_s with its canonical action. The group $H^n(G(k_s|k); \mu_\ell^{\otimes n})$ is canonically isomorphic to $H^n_{ét}(Spec(k); \mu_\ell^{\otimes n})$. Using the identifications from corollary 3.4 and lemma 4.3 the comparison homomorphism

$$H^n_B(Spec(k); \mathbf{Z}/\ell(n)) \to H^n_L(Spec(k); \mathbf{Z}/\ell(n))$$

coincides with the above homomorphism.

The following fundamental theorem is essentially a reformulation of the main results of [40] and is the starting point of Voevodsky's proof :

Theorem 4.4. *Let k be a field which admits resolutions of singularities and ℓ a prime number not equal to $char(k)$. Then for any integer $n \geq 0$ the following conditions are equivalent :*

1. *For any field extension $k \subset F$ of finite type the homomorphism*

$$h_F : K^M_n(F)/\ell \to H^n(Gal(F_s|F); \mu_\ell^{\otimes n})$$

is an isomorphism.

2. *For any smooth k-variety X and any integer $m \in \{0, .., n+1\}$, the homomorphism*

$$H^m_B(X; \mathbf{Z}_{(\ell)}(n)) \to H^m_L(X; \mathbf{Z}_{(\ell)}(n))$$

is an isomorphism.

3. *For any field extension $k \subset F$ of finite type one has :*

$$H^{n+1}_L(Spec(F); \mathbf{Z}_{(\ell)}(n)) = 0.$$

Remark 4.5. K. Kato conjectured statement 1) for any k [14]. For ℓ odd, it is only known to be true in degree 2, as the famous Merkurjev-Suslin theorem [20]. It has recently been announced for $\ell = 3$ in degrees 3 and 4 by Rost and Voevodsky. Statement 2) is known as a conjecture of S. Lichtenbaum [18].

For $n = 0$, the theorem is essentially trivial (the group $H^1_L(Spec(k); \mathbf{Z}_\ell(0))$ vanishes because any continuous homomorphism from a profinite group to a torsion-free discrete group is trivial). For $n = 1$ statement 3) means the vanishing for all F of the group $H^1(Gal(F_s|k); F_s^*) \otimes \mathbf{Z}_{(\ell)}$ (which is implied by Hilbert's theorem 90), which easily implies 1). For a general n, the implications 1) $\Rightarrow$ 2) and 3) $\Rightarrow$ 1) are nontrivial and use the main results of [40].

To prove 2) $\Rightarrow$ 3), observe that F is the union of its sub-k-algebras A_α smooth and of finite type. Then $H^{n+1}_L(Spec(F); \mathbf{Z}_{(\ell)}(n))$ is shown to be the (filtering) colimit of the groups $H^{n+1}_L(Spec(A_\alpha); \mathbf{Z}_{(\ell)}(n))$. This colimit is, by assumption, isomorphic to the colimit of the groups $H^{n+1}_B(Spec(A_\alpha); \mathbf{Z}_{(\ell)}(n))$. The latter is seen to be isomorphic to $H^{n+1}_B(Spec(F); \mathbf{Z}_{(\ell)}(n))$ which vanishes.

Definition 4.6. Let ℓ be a prime number and $n \geq 0$ an integer. We shall say that property $H90(n, \ell)$ holds if

$$H^{n+1}_L(Spec(k); \mathbf{Z}_{(\ell)}(n)) = 0$$

for any field k of characteristic 0.

Theorem 4.7. [43] *For any integer $n \geq 0$, $H90(n, 2)$ holds.*

In view of theorem 4.4, theorem 4.7 implies Milnor's conjecture for all fields of characteristic 0. But it is not difficult to deduce Milnor's conjecture in general : one uses standard relations between the Milnor K-theory (resp. Galois cohomology) of a field, complete for a discrete valuation, and of its residue field [24], [35], [43].

5. Strategy of the proof

In this section we only consider fields of characteristic zero (to ensure the resolution of singularities), and we fix a prime number ℓ.

Theorem 5.1. [43] *Let $n \geq 1$ be an integer. Assume $H90(n-1, \ell)$. Let k be a field of characteristic 0 which has no nontrivial extension of degree prime to ℓ and such that :*

$$K^M_n(k)/\ell = 0.$$

Then $H^{n+1}_L(Spec(k); \mathbf{Z}_{(\ell)}(n)) = 0$.

In the proof of this theorem, one uses the following lemma which is obviously of independent interest :

Lemma 5.2. [43] *Let $n \geq 1$. Assume $H90(n-1, \ell)$. Let k be a field of characteristic 0, without extensions of degree prime to ℓ. Let $k \subset E$ be a Galois extension of degree ℓ and σ a generator of the Galois group. Then the sequence*

$$K^M_n(E) \stackrel{1-\sigma}{\to} K^M_n(E) \stackrel{N_{E|k}}{\to} K^M_n(k)$$

is exact.

Once we have the above lemma, the deduction of theorem 5.1 only uses elementary tricks in Milnor K-theory and Galois cohomology, directly inspired by Merkurjev and Suslin's original method.

Definition 5.3. Let $\underline{a} = (a_1, \dots, a_n) \in (k^*)^n$. An *$\ell$-splitting variety* for $\underline{a}$ is a smooth, irreducible, k-variety X such that the image of the product $(a_1)..(a_n)$ in $K_n^M(k(X))/\ell$ is zero (here $k(X)$ is the function field of X). A *generic* ℓ-splitting variety for $\underline{a}$ is an ℓ-splitting variety such that it has a rational point in a given field extension $k \subset L$ if and only if the product $(a_1)..(a_n)$ vanishes in $K_n^M(L)/\ell$.

Let X be a smooth k-variety. Let us denote $\check{C}(X)$ the *simplicial smooth k-variety* which equals X^{n+1} in dimension n and with faces (resp. degeneracies) given by partial projections (resp. diagonals). It has the characteristic property that for any simplicial smooth k-variety $\mathcal{Y}$, the map $Hom_{\Delta^{op}Sm(k)}(\mathcal{Y}, \check{C}(X)) \to Hom_{Sm(k)}(\mathcal{Y}_0, X)$ is bijective. It is not difficult to extend the definitions of motivic cohomology groups to simplicial smooth k-varieties (see also 6.3 below).

Theorem 5.4. [43] *Let $n \geq 2$ be an integer. Assume $H90(n-1, \ell)$. If for any field k of characteristic 0 and any $\underline{a} = (a_1, \dots, a_n) \in (k^*)^n$ there is an ℓ-splitting variety $X_{\underline{a}}$ for $\underline{a}$ with the following properties :*

1. *the k-variety $X_{\underline{a}}$ becomes* rational *over its function field $k(X_{\underline{a}})$,*
2. $H_B^{n+1}(\check{C}(X_{\underline{a}}), \mathbf{Z}_{(\ell)}(n)) = 0$,

then $H90(n, \ell)$ holds.

Observe that the group in the statement has index B and not L.

Remark 5.5. Let Y be a smooth k-variety. If there exists a morphism $Y \to X$, then the projection $\check{C}(X) \times Y \to Y$ is a *simplicial homotopy equivalence*. For example, if X has a rational point over k, the group $H_B^{n+1}(\check{C}(X), \mathbf{Z}_{(\ell)}(n))$ is the same as $H_B^{n+1}(Spec(k), \mathbf{Z}_{(\ell)}(n))$ and thus vanishes. When the product $(a_1)\dots(a_n)$ is nontrivial in $K_*^M(k)/\ell$, an ℓ-splitting variety for $\underline{a}$ will tend, of course, to have no rational point over k.

The following lemma explains the role of the hypotheses of theorem 5.4.

Lemma 5.6. [43] *Let $n \geq 1$ be an integer. Assume $H90(n-1, \ell)$. Let k be a field of characteristic 0 and X be a smooth k-variety. Assume that :*

1. *X becomes rational over its function field $k(X)$,*
2. $H_B^{n+1}(\check{C}(X), \mathbf{Z}_{(\ell)}(n)) = 0$.

Then the homomorphism

$$H_L^{n+1}(Spec(k); \mathbf{Z}_{(\ell)}(n)) \to H_L^{n+1}(Spec(k(X)); \mathbf{Z}_{(\ell)}(n))$$

is a monomorphism.

Proof of theorem 5.4. Let $n \geq 2$ be an integer. Assume $H90(n-1, \ell)$.

Lemma 5.7. *Let F be a field of characteristic 0. Then there exists a field extension $F \subset F'$ such that F' has no finite extension of degree prime to ℓ, the homomorphism $K_n^M(F)/\ell \to K_n^M(F')/\ell$ is zero, and the homomorphism $H_L^{n+1}(Spec(F); \mathbf{Z}_{(\ell)}(n)) \to H_L^{n+1}(Spec(F'); \mathbf{Z}_{(\ell)}(n))$ is a monomorphism.*

Indeed, from lemma 5.6 and the assumption in theorem 5.4, one can find a field extension $F \subset F''$ with the last two conditions satisfied (choose a well-ordering on the set $(F^*)^n$ and proceed by transfinite induction). Then choose an algebraic extension $F'' \subset F'$ such that the absolute Galois group of F' is an ℓ-Sylow subgroup of that of F'' ; F' has no finite extension of degree prime to ℓ and a transfer argument shows that the last two conditions are preserved.

Now let k be a field of characteristic 0. Iterated applications of the lemma and taking the union yield a field extension $k \subset k^{\infty}$ such that k^{∞} has no finite extension of degree prime to ℓ, the group $K_n^M(k^{\infty})/\ell$ is zero, and the homomorphism $H_L^{n+1}(Spec(k); \mathbf{Z}_{(\ell)}(n)) \to H_L^{n+1}(Spec(k^{\infty})); \mathbf{Z}_{(\ell)}(n))$ is a monomorphism.

From theorem 5.1 the first two conditions imply the vanishing of the group $H_L^{n+1}(Spec(k^{\infty}); \mathbf{Z}_{(\ell)}(n))$ and the injectivity condition above gives

$$H_L^{n+1}(Spec(k); \mathbf{Z}_{(\ell)}(n)) = 0$$

which proves $H90(n, \ell)$. □

To find ℓ-splitting varieties with the properties required in Theorem 5.4, we have to assume now that $\ell = 2$. Let $n \geq 1$ be an integer, and $\underline{a} = (a_1, \dots, a_n) \in (k^*)^n$. Let us denote $Q_{\underline{a}}$ the (smooth) quadric in the 2^{n-1} dimensional projective space over k defined by the quadratic form (in $2^{n-1} + 1$ variables) :

$$<< a_1, \dots, a_{n-1} >> \oplus < -a_n > .$$

The symbol $<< a_1, \dots, a_{n-1} >>$ means the Pfister form on $(a_1, \dots, a_{n-1})$, *i.e.* the tensor product $(< 1 > \oplus < -a_1 >) \otimes \cdots \otimes (< 1 > \oplus < -a_{n-1} >)$ of the 2-dimensional quadratic forms $< 1 > \oplus < -a_i >$, $i \in \{1, \dots, n-1\}$, where $< a >$ means the 1-dimensional quadratic form determined by $a \in k^*$. The fact that $Q_{\underline{a}}$ is a 2-splitting variety for $\underline{a}$ follows from the theory of Pfister quadratic forms; [32], chapter 4 for instance. (In fact, $Q_{\underline{a}}$ is a generic 2-splitting variety for $\underline{a}$ following [9], [15].)

In view of theorem 5.4, the following result achieves, by induction, the proof of theorem 4.7. (We already saw that $H90(0, 2)$ and $H90(1, 2)$ are true.)

Theorem 5.8. [43] *Let $n \geq 2$ be an integer. Assume $H90(n-1, 2)$. Let k be a field of characteristic 0 and $\underline{a} = (a_1, .., a_n) \in (k^*)^n$. Then*

$$H_B^{n+1}(\check{C}(Q_{\underline{a}}), \mathbf{Z}_{(2)}(n)) = 0.$$

This theorem follows from the next two results.

Theorem 5.9. [43] *Let $n \geq 2$ be an integer. Let k be a field of characteristic 0 and $\underline{a} = (a_1, .., a_n) \in (k^*)^n$. Then*

$$H_B^{2^n-1}(\check{C}(Q_{\underline{a}}), \mathbf{Z}_{(2)}(2^{n-1})) = 0.$$

For $n = 2$ this is all that is needed. For $n \geq 3$, one needs :

Theorem 5.10. [43] *Let $n \geq 2$ be an integer and k be a field of characteristic 0. There exists a natural transformation*

$$\Theta_{\mathcal{X}}^n : H_B^{n+1}(\mathcal{X}; \mathbf{Z}/2(n)) \to H_B^{2^n-1}(\mathcal{X}, \mathbf{Z}/2(2^{n-1}))$$

on the category of simplicial smooth k-varieties $\mathcal{X}$ which preserves mod 2 reduction of integral classes and such that, if we assume $H90(n-1, 2)$, $\Theta_{\check{C}(Q_{\underline{a}})}^n$ is a monomorphism for any $\underline{a} = (a_1, \dots, a_n) \in (k^*)^n$.

Proof of theorem 5.8. As quadrics always have a rational point in some quadratic extension, a transfer argument shows that the integral motivic cohomology of $\check{C}(Q_{\underline{a}})$ is 2-torsion and thus embeds into the mod 2 motivic cohomology. In view of theorems 5.9 and 5.10 this implies the conclusion of theorem 5.8.

The proof of theorem 5.9 uses results of M. Rost [30], [31] and is sketched in the next section. The proof of theorem 5.10 uses some divisibility properties of characteristic numbers of quadrics and is sketched in section 7. □

6. The Rost motives

6.1. **The triangulated category of effective mixed motives.** The triangulated category of mixed motives over k, constructed in [45], plays for algebraic k-varieties the role that the derived category of abelian groups plays for topological spaces *via* the singular chain complex functor.

Definition 6.1. A *Nisnevich sheaf with transfers* (on $Sm(k)$) is an additive functor $F : SmCor(k)^{op} \to \mathcal{A}b$ (see section 3.1 for the definition of the category $SmCor(k)$) with the following additional property. For any cartesian square in $Sm(k)$ of the form

$$\begin{array}{ccc} U \times_X V & \to & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X \end{array}$$

such that p is an étale morphism, i is an open embedding and $p^{-1}(X - U) \to X - U$ is an isomorphism of schemes (with the reduced induced structure), the homomorphism

$$F(X) \to F(U) \times_{F(U \times_X V)} F(V)$$

is an isomorphism.

(The last condition can be shown [26] to characterize sheaves in the Nisnevich topology [27].)

The basic example of Nisnevich sheaf with transfers is the functor $L[X] : Y \mapsto c(Y, X)$ considered in section 3.1. The category $N^{tr}(k)$ of Nisnevich sheaves with transfers is abelian. Let $DN^{tr}(k)$ be its derived category. For any $n \in \mathbf{Z}$, $C_* \mapsto C_*[n]$ denotes the n-th suspension functor.

Definition 6.2. A complex of sheaves with transfers C_* is said to be $\mathbf{A}^1$-*local* if for any smooth k-variety X and any integer $n \in \mathbf{Z}$ the homomorphism

$$Hom_{DN^{tr}(k)}(L[X], C_*[n]) \to Hom_{DN^{tr}(k)}(L[X \times \mathbf{A}^1], C_*[n]),$$

induced by the projection $X \times \mathbf{A}^1 \to X$, is an isomorphism.

The triangulated category of *effective* mixed motives over k, which is denoted $DM^{eff}(k)$, is the full subcategory of DN_k^{tr} consisting of $\mathbf{A}^1$-local complexes.

The inclusion $DM^{eff}(k) \subset DN^{tr}(k)$ has a left adjoint, called the $\mathbf{A}^1$-localization, $DN^{tr}(k) \to DM^{eff}(k)$. Using it we may regard the category $DM^{eff}(k)$ as obtained from $DN^{tr}(k)$ by inverting all $\mathbf{A}^1$-quasi-isomorphisms, which means morphisms which become isomorphisms after applying the $\mathbf{A}^1$-localization functor. For any simplicial smooth k-variety $\mathcal{X}$, denote by $M[\mathcal{X}]$ the $\mathbf{A}^1$-localization of the complex of Nisnevich sheaves with transfers associated to the simplicial Nisnevich sheaf with transfers $L[\mathcal{X}]$ and call it the *motive* of $\mathcal{X}$. The formula $M[\mathcal{X}] \otimes M[\mathcal{Y}] := M[\mathcal{X} \times \mathcal{Y}]$ induces a "tensor product" $\otimes$ in $DM^{eff}(k)$.

For any smooth k-variety X, the motivic chain complex $\underline{C}_*(X)$ is $\mathbf{A}^1$-local and the morphism $L[X] \to \underline{C}_*(X)$ is an $\mathbf{A}^1$-quasi-isomorphism ; this allows one to identify $M[X]$ with the motivic chain complex $\underline{C}_*(X)$ (see [45] ; this uses [44]). For any abelian group A, any $i \geq 0$, the complexes $A(i)$ are $\mathbf{A}^1$-local as well.

Proposition 6.3. [45] *For any integers $n \in \mathbf{Z}$, $i \geq 0$, any abelian group A, and any simplicial smooth k-variety $\mathcal{X}$ the homomorphism*

$$H_B^n(\mathcal{X}; A(i)) \to Hom_{DM^{eff}(k)}(\underline{C}_*(\mathcal{X}); A(i)[n])$$

is an isomorphism.

The analogous property would be false if we used Zariski sheaves with transfers instead. This is one of the reasons why Nisnevich sheaves appear. The basic property used in the proof [45] is that any finite scheme over the spectrum of a henselian local ring (the *points* in the Nisnevich topology) is the disjoint union of the same kind of schemes.

Relation with "standard motives". It is proven in [45], [11] that for any field k of characteristic zero, any smooth k-variety X and any $n \geq 0$ the group $H^{2n}_B(X;\mathbf{Z}(n))$ is canonically isomorphic to the n-th Chow group $CH^n(X)$ of cycles of codimension n on X modulo rational equivalence. It follows that for such a field, the full subcategory of $DM^{eff}(k)$ whose objects are direct summands of motives of smooth projective k-varieties is equivalent to the category $Chow^{eff}(k)$ of effective Chow motives (with integral coefficients).

Rost's results only concern the motives of projective quadrics, which live in $Chow^{eff}(k)$. Nevertheless, in order to deduce theorem 5.9 from those results, Voevodsky has to go out of $Chow^{eff}(k)$. For instance, he uses the motives of $\check{C}(Q_{\underline{a}})$ (which are not in $Chow^{eff}(k)$ for a general $\underline{a}$) in :

6.2. **The basic distinguished triangles.**

Theorem 6.4. [43] *Let $n \geq 1$ be an integer. Let k be a field of characteristic 0 and $\underline{a} = (a_1, .., a_n) \in (k^*)^n$. Then there exist a direct summand $M_{\underline{a}}$ of $M(Q_{\underline{a}})$ and a distinguished triangle in $DM^{eff}(k)$ of the form :*

$$M[\check{C}(Q_{\underline{a}})](2^{n-1}-1)[2^n-2] \to M_{\underline{a}} \to M[\check{C}(Q_{\underline{a}})] \to M[\check{C}(Q_{\underline{a}})](2^{n-1}-1)[2^n-1].$$

The proof is based on work of M. Rost [31] in which he constructs the direct factor $M_{\underline{a}}$ of $M[Q_{\underline{a}}]$ explicitly. For any field extension $k \subset L$ in which $Q_{\underline{a}}$ has a rational point, $M[Q_{\underline{a}}|_L]$ contains as a direct summand the sum in $Chow^{eff}(k)$ of $\mathbf{Z}$ (corresponding to any rational point) and of $\mathbf{Z}(2^{n-1}-1)[2^n-2]$ (corresponding to the fundamental class). Rost's theorem is that $M[Q_{\underline{a}}]$ contains a direct factor $M_{\underline{a}}$, unique up to isomorphism, which coincides with the one described above in any field extension $k \subset L$ in which $Q_{\underline{a}}$ has a rational point.

Example 6.5. The case $n = 1$ may help to clarify the picture. The quadric $Q_{\underline{a}}$ is the spectrum of a quadratic extension L of k, $M_{\underline{a}} = M[Spec(L)]$, and the complex $M[\check{C}(Spec(L))]$ is isomorphic to the chain complex

$$\ldots \to L[Spec(L)] \to \ldots \to L[Spec(L)]$$

ending in dimension zero, whose differential is alternatively $Id - \sigma$ or $Id + \sigma$, where σ is the nontrivial k-automorphism of $Spec(L)$. The exact triangle is easy to deduce.

For $n = 2$, the quadric $Q_{\underline{a}}$ is a conic and we still have $M_{\underline{a}} = M[Q_{\underline{a}}]$. For $n \geq 3$ this is no longer true. Anyway, for general n, one has a kind of "higher dimensional Galois" phenomenon (which reduces to Galois theory for $n = 1$) : there is an isomorphism in $DM^{eff}(k)$

$$M_{\underline{a}} \otimes M_{\underline{a}} \cong M_{\underline{a}} \otimes (\mathbf{Z} \oplus \mathbf{Z}(2^{n-1}-1)[2^n-2]).$$

The proof of theorem 5.9 will also use the following result due to M. Rost :

Theorem 6.6. [30] *Let $n \geq 2$ be an integer. Let k be a field of characteristic* 0 *and $\underline{a} = (a_1, \ldots, a_n) \in (k^*)^n$. Then the canonical homomorphism*

$$H_B^{2^n-1}(Q_{\underline{a}}; \mathbf{Z}(2^{n-1})) \to k^*$$

is injective.

The group on the left hand side can be described as the cokernel of the morphism

$$\oplus_{x \in (Q_{\underline{a}})_1} K_2^M(k(x)) \to \oplus_{x \in (Q_{\underline{a}})_0} k(x)^*$$

where $(Q_{\underline{a}})_i$ means the set of points x of dimension i, and $k(x)$ is the residue field of x. The homomorphism is induced by residue homomorphisms in Milnor K-theory, and the homomorphism from the cokernel to k^* is induced by the norm maps. The proof consists, using the *geometry of Pfister quadratic forms*, of reducing to the case $n = 2$ which is due to Suslin [38].

Proof of theorem 5.9. From proposition 6.3 and theorem 6.4 we get an exact sequence

$$H_B^0(\check{C}(Q_{\underline{a}}); \mathbf{Z}_{(2)}(1)) \to H_B^{2^n-1}(\check{C}(Q_{\underline{a}}); \mathbf{Z}_{(2)}(2^{n-1})) \to H_B^{2^n-1}(M_{\underline{a}}; \mathbf{Z}_{(2)}(2^{n-1})).$$

The group on the left is easily seen to vanish. It thus suffices to prove that the homomorphism on the right is trivial. But from theorem 6.6 we conclude that the homomorphism $H_B^{2^n-1}(M_{\underline{a}}; \mathbf{Z}_{(2)}(2^{n-1})) \to H_B^{2^n-1}(M_{\underline{a}}|_{k_s}; \mathbf{Z}_{(2)}(2^{n-1}))$ is a monomorphism (since it embeds into the injection $k^* \to {k_s}^*$). Thus it suffices to prove that the homomorphism $H_B^{2^n-1}(\check{C}(Q_{\underline{a}}); \mathbf{Z}_{(2)}(2^{n-1})) \to H_B^{2^n-1}(\check{C}(Q_{\underline{a}})|_{k_s}; \mathbf{Z}_{(2)}(2^{n-1}))$ is trivial, which is now obvious since $Q_{\underline{a}}$ has a rational point in the separable closure of k so that the latter group vanishes by 5.5. □

7. Cohomology operations in motivic cohomology

In this section ℓ is a prime number and we assume $char(k) = 0$ to ensure the resolution of singularities.

7.1. **Motivic cohomology of the projective line.** Let $\mathbf{P}^1$ be the projective line over k. The canonical invertible sheaf $\mathcal{O}(1)$ on $\mathbf{P}^1$ defines a motivic cohomology class $c_1 \in H_B^2(\mathbf{P}^1; \mathbf{Z}(1))$.

Theorem 7.1. [45] *Let A be a commutative ring and $\mathcal{X}$ be a simplicial smooth k-variety. Then the $H_B^*(\mathcal{X}; A(*))$-module $H_B^*(\mathcal{X} \times \mathbf{P}^1; A(*))$ is free with basis $\{1, pr^* c_1\}$ ($pr : \mathcal{X} \times \mathbf{P}^1 \to \mathbf{P}^1$ is the projection).*

This result (which one could call the *suspension isomorphism*) follows from the following fact proven in [45] (using results on algebraic cycles cohomology from [11]) : let $DM(k)$ be the *triangulated category of mixed motives*, (more or less) formally obtained from $DM^{eff}(k)$ by inverting $\mathbf{Z}(1)$ for the $\otimes$-product in $DM^{eff}(k)$. Then the canonical functor $DM^{eff}(k) \to DM(k)$ is fully faithful.

7.2. **Basic properties of Q_i's operations.**

Definition 7.2. A (mod ℓ) *stable cohomology operation* of bidegree (p, q) is a collection Θ of natural transformations in the simplicial smooth k-variety $\mathcal{X}$:

$$\Theta^{n,i} : H_B^n(\mathcal{X}; \mathbf{Z}/\ell(i)) \to H_B^{n+p}(\mathcal{X}; \mathbf{Z}/\ell(i+q))$$

such that for any $\alpha \in H_B^n(\mathcal{X}; \mathbf{Z}/\ell(i))$, one has

$$\Theta^{n,i}(\alpha \smile c_1) = \Theta^{n-2,i-1}(\alpha) \smile c_1$$

in the group $H_B^{n+p}(\mathcal{X} \times \mathbf{P}^1; \mathbf{Z}/\ell(i+q))$.

Example 7.3. 1) The collection of all Bockstein homomorphisms $H_B^n(\mathcal{X}; \mathbf{Z}/\ell(i)) \to H_B^{n+1}(\mathcal{X}; \mathbf{Z}/\ell(i))$ defines a (mod ℓ) stable cohomology operation β of bidegree $(1,0)$ which is called the Bockstein operation.

2) Any element of $H_B^p(Spec(k); \mathbf{Z}/\ell(q))$ defines, using the product, a (mod ℓ) stable cohomology operation of bidegree (p,q).

In [47], Voevodsky constructs for any $i \geq 0$, a (mod ℓ) stable cohomology operation of bidegree $(2(\ell-1)i, (\ell-1)i)$, denoted P^i and called the i-th *reduced power operation*, by adapting the standard constructions from algebraic topology [37]. These P^i's altogether are characterized by the following properties :

1. $P^0 = Id$.
2. For any simplicial smooth k-variety $\mathcal{X}$ and any $u \in H_B^n(\mathcal{X}; \mathbf{Z}/\ell(i))$ one has $P^i(u) = 0$ for $n < 2i$ and $P^i(u) = u^\ell$ for $n = 2i$.
3. *(Cartan formula)* For any simplicial smooth k-varieties $\mathcal{X}$ and $\mathcal{Y}$ and any $u \in H_B^n(\mathcal{X}; \mathbf{Z}/\ell(p))$, $v \in H_B^m(\mathcal{Y}; \mathbf{Z}/\ell(q))$ one has
$$P^i(u \otimes v) = \Sigma_{a+b=i} P^a(u) \otimes P^b(v) + \tau(\Sigma_{a+b=i-2} \beta P^a(u) \otimes \beta P^b(v)),$$
where τ denotes the nonzero element of $H_B^0(Spec(k); \mathbf{Z}/2(1)) \cong \mu_2(k)$ if $\ell = 2$ and is zero if $\ell \neq 2$.

Definition 7.4. Set $Q_0 = \beta$. For $i \geq 1$ define inductively :
$$Q_i = [Q_{i-1}, P^{\ell^i}]$$
(the notation $[-,-]$ means the commutator of two operations).

The operation Q_i has bidegree $(2\ell^i - 1, \ell^i - 1)$.[2] The following result follows from the description of the algebra of all (mod ℓ) stable cohomology operations in terms of the motivic cohomology of $Spec(k)$, the Bockstein, and the P^i's [47] :

Theorem 7.5. [43, 47] *Let k be a field of characteristic zero. Then*

1. *for any $i \geq 0$ one has $Q_i^2 = 0$;*
2. *for any $i \geq 0$, there is a canonical (mod ℓ) stable cohomology operation q_i such that $Q_i = [\beta, q_i]$.*

7.3. **v_n-varieties.**

The characteristic class s_d. Let d be an integer ≥ 0. Recall from [25] that there is a characteristic class s_d for complex vector bundles over CW-complexes with value in singular cohomology with integral coefficients having the following characteristic properties :

1. for any line bundle $\mathcal{L}$, $s_d(\mathcal{L}) = c_1(\mathcal{L})^d$ (d-th power of the first Chern class) ;
2. for any two vector bundles (over the same base) E and F, $s_d(E \oplus F) = s_d(E) + s_d(F)$.

(s_d is the characteristic class corresponding to the d-th Newton polynomial in the Chern classes c_i's.) We still write s_d for the characteristic class of algebraic vector bundles over smooth k-varieties given by the same polynomial in the Chern classes with values in the Chow groups of smooth k-varieties [12]. A characteristic number

[2] Through any complex embedding of k, the P^i's are compatible with the *reduced power operations* P^i's of [37] and the Q_i's to Milnor's elements [23].

of a smooth proper k-variety X of pure dimension d is the degree of an element in $CH^d(X)$ given by a polynomial (with integral coefficients) in the Chern classes of the normal bundle ν_X (the opposite of the tangent bundle in $K_0(X)$).

Definition 7.6. Let $n \geq 0$ be an integer and X be a smooth proper k-variety of pure dimension $\ell^n - 1$. We say that X is a *v_n-variety* (at ℓ) if all the characteristic numbers of X are divisible by ℓ, and if the characteristic number $s_{\ell^n-1}(X)$ is not divisible by ℓ^2.

For $n = 0$ a v_0-variety (at ℓ) is a 0-dimensional smooth k-variety whose degree over k is divisible by ℓ and not by ℓ^2.

Remark 7.7. The characteristic number $s_{\ell^n-1}(X)$ is always divisible by ℓ for $n \geq 1$. To prove this it is sufficient to consider the case $k \subset \mathbf{C}$, which follows from [1], Part II, lemma 7.9(iii) and theorem 8.2. Recall also from the same reference that the class of an almost complex, compact, differentiable manifold X of dimension $\ell^n - 1$ in the complex cobordism ring localized at ℓ, $MU_* \otimes \mathbf{Z}_{(\ell)}$, is a multiplicative generator of this ring if and only if $s_{\ell^n-1}(X)$ is not divisible by ℓ^2.

Remark 7.8. Let $n \geq 0$ and H be a smooth hypersurface of degree d in $P_k^{\ell^n}$. From [25], Problem 16-D, the characteristic number $s_{\ell^n-1}(H)$ equals $-d(1 - d^{\ell^n-1} + \ell^n)$ (the sign comes from the fact that we used the normal bundle rather than the tangent bundle ; as s_{ℓ^n-1} is additive, this only changes the sign. Observe also this number is divisible by ℓ if $n > 0$ by Fermat's little theorem). As the other characteristic numbers are obviously divisible by d, we see that H is a v_n-variety (at ℓ) if d is divisible by ℓ and not by ℓ^2.

7.4. **On the vanishing of some Margolis homology groups.**

Definition 7.9. Let $\mathcal{X}$ be a pointed simplicial smooth k-variety. We denote $\tilde{H}^*(\mathcal{X}; \mathbf{Z}/\ell(*))$ the kernel of the homomorphism (induced by the base point) : $H_B^*(\mathcal{X}; \mathbf{Z}/\ell(*)) \to H_B^*(Spec(k); \mathbf{Z}/\ell(*))$ and call it the *reduced motivic cohomology* with $\mathbf{Z}/\ell$-coefficients of $\mathcal{X}$. For any $i \geq 0$ the operation Q_i acts on this bigraded abelian group which can thus be considered as a complex (recall from 7.5 that $Q_i^2 = 0$), and we denote $HM_i^*(\mathcal{X}; \mathbf{Z}/\ell(*))$ its homology and call it the i-th Margolis homology of $\mathcal{X}$ (at ℓ).

For any morphism of simplicial smooth k-varieties $f : \mathcal{X} \to \mathcal{Y}$ one defines by the usual formulas the *cone* of f, denoted $C(f)$. It is characterized by the property that morphisms from it to any simplicial smooth k-variety $\mathcal{Z}$ is in bijection with the set of morphisms $\mathcal{Y} \to \mathcal{Z}$ together with a *simplicial homotopy* to the trivial morphism of its composition with f. This is a pointed simplicial smooth k-variety. We have long exact sequences relating motivic cohomology groups of the source, the target and the cone of f.

Definition 7.10. Let X be a smooth k-variety and $n \in \mathbf{N}$ an integer. A *v_n-point* of X is a morphism $Y \to X$ with Y a v_n-variety (at ℓ).

Remark 7.11. It should be possible to show that any v_n-variety has v_m-points for $m \in \{0, .., n\}$.

Theorem 7.12. [43] *Let X be a smooth k-variety and $\underline{C}(X)$ be the cone of the morphism $\check{C}(X) \to Spec(k)$. Assume that X has a v_n-point for some $n \in \mathbf{N}$. Then the Margolis homology $HM_n^*(\underline{C}(X); \mathbf{Z}/\ell(*))$ vanishes.*

Proof of theorem 5.10. Let Θ_k^n be the composition $Q_{n-2} \circ .. \circ Q_1$. The second property in theorem 7.5 implies that this operation preserves reduction of integral classes. Let us prove the required injectivity of Θ_k^n. Let $i \in \{1, .., n-3\}$ and assume that the composition $Q_i \circ .. \circ Q_1$ from $H_B^{n+1}(\check{C}(Q_{\underline{a}}), \mathbf{Z}_{(2)}(n))$ is injective. Then the operation Q_{i+1} is injective on the target of the previous operation.

Indeed, the motivic cohomology groups of $\check{C}(Q_{\underline{a}})$ coincide in the relevant bidegrees with the obvious shift of that of $\underline{C}(Q_{\underline{a}})$. So it suffices to prove injectivity of Q_{i+1} on the corresponding motivic cohomology group of $\underline{C}(Q_{\underline{a}})$.

But the quadric $Q_{\underline{a}}$ of dimension $2^{n-1}-1$ has v_m-points for any $m \in \{0, .., n-1\}$: take any closed subquadric of the correct dimension and use 7.8. Thus, we deduce from theorem 7.12 that the kernel of the operation Q_{i+1} considered above is covered by the image of the previous Q_{i+1}. But the source of the latter vanishes by the inductive assumption that $H90(n-1, 2)$ holds. □

To complete this sketch of the proof of theorem 2.2 it only remains to give a :

Proof of Theorem 7.12. Observe that the analogue of this theorem with étale motivic cohomology groups is trivial because the reduced mod ℓ étale motivic cohomology of $\underline{C}(X)$ vanishes (X has a rational point in a separable extension of k) as do all the Margolis homology. In much the same way, if X has a rational point in a finite extension of k of degree prime to ℓ, then the reduced mod ℓ motivic cohomology of $\underline{C}(X)$ vanishes (use a transfer argument) and all the Margolis homology do as well.

Let us first prove the case $n = 0$ of theorem 7.12. The conclusion means that the homology of the bigraded group $\tilde{H}_B^*(\underline{C}(X); \mathbf{Z}/\ell(*))$ with respect to the Bockstein $\beta = Q_0$ vanishes. This vanishing is equivalent to the fact that the multiplication by ℓ : $\tilde{H}_B^*(\underline{C}(X); \mathbf{Z}/\ell^2(*)) \to \tilde{H}_B^{*+1}(\underline{C}(X); \mathbf{Z}/\ell^2(*))$ is trivial (by the long exact sequences associated to $0 \to \mathbf{Z}/\ell \to \mathbf{Z}/\ell^2 \to \mathbf{Z}/\ell \to 0$). On the other hand, the assumption on X means that it has a rational point in a finite extension L of k of degree divisible by ℓ and not by ℓ^2, *i.e.* of the form $\ell.m$ with m prime to ℓ. The composition of the morphism $\pi^* : \tilde{H}_B^*(\underline{C}(X); \mathbf{Z}/\ell^2(*)) \to \tilde{H}_B^*(\underline{C}(X|L); \mathbf{Z}/\ell^2(*))$ (induced by the projection $\pi : Spec(L) \to Spec(k)$) and the transfer morphism $\pi_* : \tilde{H}_B^*(\underline{C}(X|L); \mathbf{Z}/\ell^2(*)) \to \tilde{H}_B^*(\underline{C}(X); \mathbf{Z}/\ell^2(*))$ is zero because the middle group is zero (as $X|L$ has a rational point). As this composition is also multiplication by the degree $\ell.m$ of the extension and as m is prime to ℓ, multiplication by ℓ is zero as well.

Roughly speaking, the general case $n \geq 1$ proceeds in a quite analogous way. Here are the changes.

1) In the above argument the motivic cohomology theory with $\mathbf{Z}/\ell^2$-coefficients appears in the problem as the fiber in the category $DM(k)$ of the morphism $\mathbf{Z}/\ell \to \mathbf{Z}/\ell[1]$ corresponding to the Bockstein. One would like to introduce for any $n \geq 0$ the fiber corresponding to the operation Q_n. Unfortunately, the group $Hom_{DM(k)}(\mathbf{Z}/\ell, \mathbf{Z}/\ell(\ell^n - 1))[2\ell^n - 1])$ is zero for $n > 0$. To solve this problem, one works in a "nonlinear" analogue of the triangulated category $DM(k)$:

2) *The stable homotopy category of smooth k-varieties*, denoted $\mathcal{SH}(k)$. Let us sketch its construction. One first localizes the category of pointed simplicial Nisnevich sheaves of *sets* on $Sm(k)$ with respect to the notion of $\mathbf{A}^1$-weak equivalences, in much the same way one defines $DM^{eff}(k)$ as a localization of the category of chain complexes of Nisnevich sheaves with transfers. We get the homotopy category

of pointed smooth k-varieties (see [26]). One then applies the standard procedure of stabilization [1], the pointed 2-sphere being replaced here by the pointed projective line over k. A *spectrum* (over k) is a sequence of pointed simplicial Nisnevich sheaves of sets $\{E_n\}_{n\geq 0}$ together with structure morphisms $\sigma_n : E_n \wedge \mathbf{P}^1 \to E_{n+1}$. Here, the notation $\mathcal{X} \wedge \mathcal{Y}$ means the smash-product of two pointed simplicial Nisnevich sheaves of sets $\mathcal{X}$ and $\mathcal{Y}$ defined as the quotient of the product $\mathcal{X} \times \mathcal{Y}$ by the union of $\mathcal{X} \times pt$ and $pt \times \mathcal{Y}$. The notion of *stable* $\mathbf{A}^1$*-weak equivalences* between spectra is derived in a standard way from that of $\mathbf{A}^1$-weak equivalences. The stable homotopy category is obtained by inverting stable $\mathbf{A}^1$-weak equivalences in the category of spectra (over k). It is a triangulated category (for which we denote as usual by $E \mapsto E[n]$ the n-th suspension functor, $n \in \mathbf{Z}$) equipped with an internal tensor product $\wedge$ induced by the *smash-product*.

Let $\Delta^{op}Sm(k)_\bullet$ be the category of pointed simplicial smooth k-varieties. For any simplicial smooth k-variety $\mathcal{X}$, we denote $\mathcal{X}_+$ the pointed simplicial smooth k-variety which equals, in dimension n, the disjoint union of $\mathcal{X}_n$ and $Spec(k)$ (the base point). There is a functor, called the *suspension spectrum* functor :

$$S(-) : \Delta^{op}Sm(k)_\bullet \to \mathcal{SH}(k), \mathcal{X} \mapsto S(\mathcal{X}),$$

such that $S(\mathcal{X}_+) \wedge S(\mathcal{Y}_+)$ is naturally isomorphic to $S((\mathcal{X} \times \mathcal{Y})_+)$. The spectrum $S^0 := S(Spec(k)_+)$ is the unit for the smash-product and the spectrum $S(\mathbf{P}^1)$ is invertible (for the smash product). Let us denote $S^0(1) := S(\mathbf{P}^1)[-2]$ and for any spectrum E and integer $i \in \mathbf{Z}$, let us denote $E(i)$ the spectrum $E \wedge (S^0(1))^{\wedge i}$. For any two spectra E and F, let us set $[E, F] := Hom_{\mathcal{SH}(k)}(E, F)$ and for any integers $n, i \in \mathbf{Z}$ set $\tilde{F}^n(E)(i) := [E, F(i)[n]]$.

The functor $S(-)$ above refines the functor M of 6.1 : there is a functor $M(-) : \mathcal{SH}(k) \to DM(k)$, such that $M(S(\mathcal{X}_+))$ is canonically isomorphic to $M[\mathcal{X}]$, for any simplicial smooth k-variety $\mathcal{X}$. The functor $M(-)$ has a right adjoint $H : DM(k) \to \mathcal{SH}(k)$ so that motivic cohomology with A coefficients is represented in $\mathcal{SH}(k)$ by a spectrum HA using the formula

$$H^n(\mathcal{X}; A(i)) \cong [S(\mathcal{X}_+); HA(i)[n]].$$

The suspension spectrum of a simplicial smooth k-variety does keep more information than just the motive : for example the action of the reduced power operations on mod ℓ motivic cohomology. Indeed, the stable homotopy category has the expected property that the group of stable mod ℓ cohomology operations of bidegree (p, q) is canonically in bijection with the group of morphisms $[H\mathbf{Z}/\ell, H\mathbf{Z}/\ell(q)[p]]$ ([47]).

3) For any $n \geq 0$ let Φ_n be the fiber of the morphism

$$Q_n : H\mathbf{Z}/\ell \to H\mathbf{Z}/\ell(\ell^n - 1)[2\ell^n - 1].$$

It is easy to check that the conclusion of theorem 7.12 is equivalent to the fact that the (obvious) composition

$$\eta_n : \Phi_n(\ell^n - 1)[2(\ell^n - 1)] \to H\mathbf{Z}/\ell(\ell^n - 1)[2(\ell^n - 1)] \to \Phi_n$$

induces the zero homomorphism

$$\tilde{\Phi}_n^{*+2(\ell^n-1)}(\underline{C}(X))(* + \ell^n - 1) \to \tilde{\Phi}_n^*(\underline{C}(X))(*).$$

4) For any vector bundle ξ over a smooth k-variety, define its *Thom* space $Th(\xi) := E(\xi)/E(\xi)^*$ to be the quotient sheaf in the Nisnevich topology of the total space of ξ by the complement of the zero section. For any integers $n, r \geq 0$

let $Gr_{n,r}$ be the Grassmanian variety classifying rank n vector bundles with $n+r$ sections generating it. Let $Th_{n,r}$ be the Thom space of the canonical rank n vector bundle over $Gr_{n,r}$ and let Th_n be the colimit of the $Th_{n,r}$ as r increases (a pointed Nisnevich sheaf). It is possible to make the collection of Th_n into a spectrum which we denote $MG\ell$.[3]

Let $U : MG\ell \to H\mathbf{Z}$ be the morphism in the stable homotopy category which corresponds to the Thom class $\in \tilde{H}^0_B(MG\ell; \mathbf{Z}(0))$, and $\overline{U} : MG\ell \to H\mathbf{Z}/\ell$ its mod ℓ reduction.

For the proof of the case $n = 0$ of theorem 7.12, we used a transfer argument. For $n \geq 1$, one uses Gysin morphisms. First, the spectrum Φ_n can be *oriented* in the following sense. There exists a morphism $\theta : MG\ell \wedge \Phi_n \to \Phi_n$ which makes the following diagram commutative in $\mathcal{SH}(k)$:

$$\begin{array}{ccc} MG\ell \wedge \Phi_n & \stackrel{\theta}{\to} & \Phi_n \\ \downarrow & & \downarrow \\ MG\ell \wedge H\mathbf{Z}/\ell & \stackrel{\theta'}{\to} & H\mathbf{Z}/\ell \end{array}$$

where θ' is the composition of the morphism (induced by $\overline{U}$) $MG\ell \wedge H\mathbf{Z}/\ell \to H\mathbf{Z}/\ell \wedge H\mathbf{Z}/\ell$ and the morphism $H\mathbf{Z}/\ell \wedge H\mathbf{Z}/\ell \to H\mathbf{Z}/\ell$ (corresponding to the product in mod ℓ motivic cohomology). The existence of such a θ is related to a property of the 'coproduct' of Q_n; see [43], lemma 3.23.

5) For any smooth proper k-variety Y of dimension d, there is a canonical morphism $[Y] : (\mathbf{P}^1)^{\wedge d} \to MG\ell \wedge S(Y_+)$ (from now on, $\mathbf{P}^1$ means the suspension spectrum of the pointed projective line), called the *fundamental class* of Y, whose definition uses the local behaviour of smooth k-varieties in the Nisnevich topology : any closed point $y \in Y \in Sm(k)$ "locally" looks like a closed point in an affine space $\mathbf{A}^d_k$. This is false in the Zariski topology.

For any spectrum E, one defines Gysin homomorphisms (depending on the choice of θ) :

$$[Y] \cap - : \tilde{\Phi}^{*+2.d}_n(S(Y_+) \wedge E)(* + d) \to \tilde{\Phi}^*_n(E)(*)$$

as the *cap-product* by the fundamental class of Y.

Assume now that $d = \ell^n - 1$. The composition of the morphism (induced by the projection $Y \to Spec(k)$)

$$\tilde{\Phi_n}^{*+2(\ell^n-1)}(E)(* + \ell^n - 1) \to \tilde{\Phi_n}^*(E \wedge S(Y_+))(*)$$

and the Gysin homomorphism above defines a natural transformation (in E) :

$$\tilde{\Phi}^{*+2(\ell^n-1)}_n(E)(* + \ell^n - 1) \to \tilde{\Phi}^*_n(E)(*)$$

which corresponds (taking $E = \Phi_n$) to an element $[Y]_{\Phi_n}$ of the group $[\Phi_n(\ell^n-1)[2(\ell^n-1)], \Phi_n]$: it is the composition of the morphism $(\mathbf{P}^1)^{\wedge \ell^n - 1} \wedge \Phi_n \to MG\ell \wedge \Phi_n$ induced by the fundamental class of Y and the orientation θ. This group can be computed : it is cyclic of order ℓ with generator η_n. Thus there is a mod ℓ integer $\lambda^Y_{\Phi_n}$ with $[Y]_{\Phi_n} = \lambda^Y_{\Phi_n}.\eta_n$.

If there is a morphism of smooth k-varieties $Y \to X$, the homomorphism

$$\lambda^Y_{\Phi_n}.\eta_n : \tilde{\Phi}^{*+2(\ell^n-1)}_n(\underline{C}(X))(* + \ell^n - 1) \to \tilde{\Phi}^*_n(\underline{C}(X))(*)$$

[3] The cohomology theory $MG\ell^*(-)(*)$ associated to the spectrum $MG\ell$ admits a multiplicative structure and a theory of Chern classes. It can be shown to be *the universal* such cohomology theory on the category of simplicial smooth k-varieties.

is zero since it factors through a cohomology group of the spectrum $\underline{C}(X) \wedge S(Y_+)$ which is trivial (because the morphism $\check{C}(X) \times Y \to Y$ is a simplicial homotopy equivalence by 5.5).

To finish the proof of theorem 7.12, it thus suffices to show that $\lambda^Y_{\Phi_n} \neq 0 \in \mathbf{Z}/\ell$ if Y is a v_n-variety, which is clearly equivalent to proving that $[Y]_{\Phi_n}$ is nonzero for any v_n-variety Y.

6) Let $\alpha : S^0 \to \Phi_n$ be the unique morphism which lifts $S^0 \to H\mathbf{Z}/\ell$ (the unit), $\psi : MG\ell \to \Phi_n$ be the composition $\theta \circ Id_{MG\ell} \wedge \alpha$, and $[Y]_{MG\ell} : (\mathbf{P}^1)^{\wedge \ell^n - 1} \to MG\ell$ be the composition of the fundamental class of Y and the projection to the point $MG\ell \wedge S(Y_+) \to MG\ell$. The fact that $[Y]_{\Phi_n} \neq 0$ follows from :

Lemma 7.13. [43] *For any v_n-variety Y over a field k of characteristic zero, one has*

$$\psi \circ [Y]_{MG\ell} \neq 0 \in [(\mathbf{P}^1)^{\wedge \ell^n - 1}, \Phi_n]. \qquad \square$$

8. Further results and comments

Applications of theorem 2.2. Using his joint work with Suslin [40], Voevodsky deduces that the Beilinson-Lichtenbaum conjecture [3], [18] on the comparison between Zariski and étale motivic cohomology of algebraic varieties holds in characteristic zero for $\mathbf{Z}/2$-coefficients. The integral version is statement 2) of theorem 4.4. Using the spectral sequence constructed by Bloch and Lichtenbaum [7], Voevodsky derives the Lichtenbaum-Quillen conjecture at 2 for fields of characteristic zero.

Milnor's conjecture on the graded ring of the Witt ring. V. Voevodsky has announced a proof of Milnor's conjecture on the graded ring associated to the Witt ring $W(k)$ of quadratic forms over k modulo hyperbolic forms [24]. This will appear in a joint paper with D. Orlov and A. Vishik [28].

Final comments. As we saw above, the proof of Milnor's conjecture described here relies on the use of new cohomology theories for simplicial smooth k-varieties which do not come from the theory of mixed motives. The first such examples were Grothendieck's K_0 and Quillen's K_n functor(s) : their restriction to the category of smooth projective k-varieties cannot be extended to functors from the category of effective *integral* Chow motives $Chow^{eff}(k)$. As is well known, the obstruction disappears when one considers rational coefficients, as Grothendieck's K_0 rationally splits as the sum of the rational Chow groups. As opposed to ordinary cohomology theories (such as motivic cohomology, étale cohomology), the theories such as algebraic K-theory, $\Phi^*_n(-)(*)$, and $MG\ell^*(-)(*)$ are called *generalized cohomology theory* in the spirit of algebraic topology.

Besides, of course, giving a spectacular proof of Milnor's conjecture, we may say as a tentative conclusion that Voevodsky's work suggests that the use of generalized cohomology theories should play an important role in a future better understanding of the category of smooth varieties over a field.

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