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Leçons sur le Théorème de Beurling et Malliavin, by Paul Koosis, Les Publications CRM Montréal, 1996, xii+230, \$33.65, ISBN 2-921120-19-4

This excellent book is warmly recommended to all who have any interest in hard analysis. Using BM as an abbreviation for Beurling-Malliavin, we begin by explaining what the BM theorem is and why it is important. Let E denote the class of entire functions $f \not\equiv 0$ of exponential type. The BM theorem addresses the following question: Given $W : \mathbf{R} \to \mathbf{R}$, under what conditions does there exist $f \in E$, of arbitrarily small type, such that f and fW are bounded on the real axis? Such a function f is called a *multiplier*, and the problem of finding it is the *multiplier problem*.

To describe the solution [2], set

$$J^-(W) = \int_{-\infty}^{\infty} \frac{\log^- |W(x)|}{1+x^2}\, dx, \qquad J^+(W) = \int_{-\infty}^{\infty} \frac{\log^+ |W(x)|}{1+x^2}\, dx.$$

Since $J^-(f) > -\infty$, the condition $J^+(W) < \infty$ is *necessary.* The BM theorem gives two supplementary hypotheses under which this necessary condition is also sufficient:

1. $W(x) = |F(x)|$ where $F \in E$.
2. $W(x) \geq 1$ and $\log W$ is uniformly Lipschitzian on $\mathbf{R}$.

In Chapter IV of his book Koosis shows that each result (1), (2) implies the other; this surprising fact seems to have been long unsuspected. The proof involves several interesting ideas, among them a construction similar to that used by F. Riesz in proving the rising-sun lemma.

Here is one application of the BM theory. Let $1 \leq p \leq \infty$, let $a > 0$, and let $B(a, p)$ be the class of functions of form

$$F(z) = \int_{-a}^{a} e^{izt} g(t)\, dt, \quad g \in L^p, \quad \|g\|_p > 0.$$

Then the following classes of entire functions F are identical:

1. Those of exponential type with $J^+(F) < \infty$.
2. Those of form F_b/F_a where $F_a \in B(a,p)$ and $F_b \in B(b,p)$ for some b.

As is said in [2], this is "a formal analog of a theorem of Nevanlinna stating that a meromorphic function with bounded characteristic in the unit disk can be expressed as the quotient of two analytic functions."

Perhaps the most important application is the solution [1], [3] of a completeness problem that had been open at least since 1916. Let $\lambda = \{\lambda_n\}$ be a sequence of positive numbers with counting function $\Lambda(u) =$ number of $\lambda_n \leq u$. Let C denote the class of sequences $\{x_n, y_n\}$ such that $(x_n, x_n + y_n)$ are nonoverlapping intervals, $x_n > 0$, and

$$\sum \left(\frac{y_n}{x_n}\right)^2 = \infty.$$

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The BM density $D(\lambda)$ is defined by

$$D(\lambda) = \sup_{C} \liminf_{n \to \infty} \frac{\Lambda(x_n + y_n) - \Lambda(x_n)}{y_n}.$$

The definition extends in an obvious way to negative sequences, and the larger of these two values is taken as $D(\lambda)$ for any real sequence λ. If $\sum |\operatorname{Im}/\lambda_n| < \infty$, the extension to complex λ is made by setting $D(\lambda) = D(\tilde{\lambda})$ where, following Levinson, $1/\tilde{\lambda}_n = \operatorname{Re} 1/\lambda_n$. Otherwise $D(\lambda) = \infty$. The assumption $\lambda_n \neq 0$ implied by this definition does no harm.

The BM theory shows that the completeness radius of $\{\exp i\lambda_n x\}$ is $\pi D(\lambda)$. That is, the set is complete L^p on every interval of length $< 2\pi D(\lambda)$ and on no larger interval. In the course of the proof the authors obtain a remarkable theorem on the class E_a of functions $F \in E$ of type a for which $J^+(F) < \infty$. Namely, if $a > \pi D(\lambda)$, then E_a contains a function vanishing on λ and if $a < \pi D(\lambda)$, then E_a contains no such function.

The above result forms a counterpart to an important theorem of Levinson [6], as completed and improved in Chapter I of Koosis' book. For $f \in E$ denote by $N^+(r, \delta)$ the counting function for zeros in a sector of angle δ with apex at the origin and centered on the positive real axis, by $N^-(r, \delta)$ the counting function for a corresponding mirror-image sector centered on the negative real axis, and by $N(r, \delta)$ the counting function for the zeros z_k in the two sectors complementary to these. Introduce the following conditions:

(a) $$\lim_{r \to \infty} \frac{N^+(r, \delta)}{r} = D, \quad \lim_{r \to \infty} \frac{N(r, \delta)}{r} = 0, \quad \lim_{r \to \infty} \frac{N^-(r, \delta)}{r} = D,$$

(b) $$\sum \Big| \operatorname{Im} \frac{1}{z_n} \Big| < \infty, \quad \lim_{r \to \infty} \sum_{|z_k| < r} \operatorname{Re} \frac{1}{z_k} \text{ exists.}$$

(c) $$\limsup_{y \to \infty} \frac{\log |f(iy)|}{y} + \limsup_{y \to \infty} \frac{\log |f(-iy)|}{y} = 2\pi D,$$

(d) $$\limsup_{x \to \infty} \frac{\log |f(x)|}{x} + \limsup_{x \to \infty} \frac{\log |f(-x)|}{x} = 0,$$

(e) $$\lim_{R \to \infty} \int_1^R \frac{\log |f(x) f(-x)|}{x^2} \, dx \text{ exists.}$$

Then Koosis' improvement of Levinson's theorem asserts that (ab) hold for all $\delta \in (0, \pi)$ if and only if (cde) hold. In broad outline Koosis follows Levinson, but the analysis is enriched by his observation that Levinson's estimates are, in essence, reversible. Departing from Levinson, Koosis also shows that a certain Tauberian argument, introduced for even f by Paley and Wiener, applies to the general case. In my view Chapter I makes a significant contribution to the history of analysis and to analysis itself.

Chapter II considers functions $f \in E$ with $J^+(f) < \infty$. It begins with the familiar fact that $J^-(f) > -\infty$. Earlier proofs depend on a rather elaborate formula of Carleman, but Koosis gives a novel and elegant proof depending on the simplest form of the Poisson formula in a half plane. He then uses a version of Jensen's theorem over a translated ellipse to study the distribution of zeros. The final result is what aficionados of the subject call *the easier half* of the BM theory. (Here I should have preferred an earlier version of this argument that uses the Koosis-Jensen formula only over a translated circle.) Next Koosis gives a fundamental lemma on the Poisson integral, due to BM, which he calls the "pierre angulaire" of his exposition. Unfortunately limitations of space preclude giving the statement here, nor is it possible to describe the two ways (original with Koosis so far as I know) in which it is used.

Chapter III contains a profound study of subharmonic functions and their majorants. It begins with a method of Hörmander which, Koosis says, "va nous permettre d'éviter une construction fastidieuse." After presenting a theorem of Lelong and Gruman (1986) he comes to the main goal: the *little multiplier theorem* of Khabibullin. The gist is that if $a > D(\lambda)$, there exists a function $f \in E_a$ with $J^+(f) < \infty$ that vanishes on λ; this is essentially one of the BM theorems mentioned above. The researches of Khabibullin (1991–1995) are scattered over numerous papers, mostly in Russian, and it is a service of the book to have made them available to a wider audience.

In Chapter V the equivalence developed in Chapter IV is combined with the results of Chapter III to get the BM multiplier theorem. Instead of giving details we mention a different version [5] that is in somewhat the same spirit but forms an interesting contrast to the argument given in the book. A theorem of Koosis (1966) that I have long admired goes as follows: Let $P(z)$ be any even polynomial such that $P(0) = 1$. Then there are absolute positive constants a, b, independent of P, such that

$$\frac{\log|P(z)|}{|z|} \le a \sum_{m=1}^{\infty} \frac{\log^+|P(m)|}{m^2}$$

provided the sum on the right is less than b. Later Koosis dropped the restriction of evenness and extended the theorem accordingly. His student Pedersen found a simpler proof [7], but his work does not by any means reduce the result to a triviality. In a *tour de force* of ingenious ideas, Koosis [5] uses this together with a result of de Branges to get the BM theorem. Further discussion of topics related to the BM theory is in [4], [8].

We conclude with a few personal impressions. In writing on Franz Liszt, the pianist Hallé expressed astonishment at "the crystal clearness which never failed him for a moment." A similar characterization applies to the book under review.— The book is loaded with novel ideas, but Koosis writes in a self-effacing style that may lead readers to underestimate his originality.— French is ideally suited for mathematical exposition, *vide* Cauchy, Goursat, Darboux, Picard, Poincaré, and a host of others. So Koosis is in good company, despite the apology on page xi.— Koosis' terminology for an important class of functions gives extreme emphasis to a mathematician that in fact had little to do with the main lines of development. He eliminates Levinson in this connection by referring to his book (which appeared five years after his papers) while for everyone else he refers to the original papers. This procedure does not have my approval.

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Raymond M. Redheffer
University of California, Los Angeles
E-mail address: rr@math.ucla.edu