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Galois module structure, by Victor P. Snaith, Fields Institute Monographs, vol. 2, American Mathematical Society, Providence, RI, 1994, vii+207 pp., \$70.00, ISBN 0-8218-0264-X

The subject of arithmetic Galois module structure is concerned with the study of the structure of various naturally occurring objects as modules over group rings of Galois groups. The simplest example of this is the normal basis theorem, which is usually encountered in a first course on Galois theory. Suppose that L/K is a finite Galois extension with Galois group G. The normal basis theorem asserts that L is a free KG-module of rank one. In particular, this implies that L is a free $\mathbb{Q}G$-module.

Now let O_L denote the ring of integers of L. It is natural to try and generalise the example afforded by the normal basis theorem by asking what one can say about the structure of O_L as a $\mathbb{Z}G$-module. This turns out to be a much deeper question. The basic starting point of this aspect of the theory is a theorem of E. Nöether ([N]) which asserts that O_L is a locally free $\mathbb{Z}G$-module if and only if the extension L/K is tamely ramified. Hence O_L determines a class (O_L) in the locally free classgroup $Cl(\mathbb{Z}G)$ of $\mathbb{Z}G$, and one is interested in understanding this class in terms of the arithmetic of the extension L/K. During the 1970's, A. Fröhlich, Ph. Cassou-Noguès, J. Martinet, M. J. Taylor and others developed a very complete theory of the structure of O_L as a $\mathbb{Z}G$-module in the tame case. The main result of this theory is quite striking. For any character χ of G, there is an extended Artin L-function $\Lambda(s, \chi)$ attached to the extension L/K. This L-function satisfies a functional equation of the form $\Lambda(1-s, \chi) = W(\chi)\Lambda(s, \chi^*)$. Here χ^* denotes the contragradient character of χ, and $W(\chi)$ is a constant called the Artin root number attached to χ. If χ is a symplectic character of G, then it is not hard to show that $W(\chi) = \pm 1$. By using the theory of locally free classgroups developed by Fröhlich, one may define a class $W_{L/K} \in Cl(\mathbb{Z}G)$ (the so–called analytic root number class) in terms of the Artin root numbers attached to symplectic characters of G. It was conjectured by Fröhlich that $(O_L) = W_{L/K}$ in $Cl(\mathbb{Z}G)$ when the extension L/K is tame. This remarkable conjecture therefore implies that the Galois module structure of O_L is determined by the symplectic Artin root numbers attached to the extension L/K. Fröhlich's conjecture was proved by M. Taylor in 1981 (see [T]). For a complete account of the tame theory up until 1983, we refer the reader to [F1] and [F2]. Further developments up until about 1990 are also reported on in [BB].

If we now assume that the extension L/K is wildly ramified, then we are faced with a very different situation. Here the ring of integers O_L is not locally free (by the result of Noether mentioned above), and so it does not determine an element in $Cl(\mathbb{Z}G)$ in the obvious way. There are a number of approaches towards circumventing this difficulty. One such approach was developed by T. Chinburg in the mid 1980's (see [C1], [C2]). For any Galois extension L/K of number fields,

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he showed how to use the canonical classes arising from the Artin-Tate cohomological approach to class field theory to define elements $\Omega(L/K,3)$ $(i=1,2,3)$ in $Cl(\mathbb{Z}G)$. The precise description of these elements is too complicated to be given here and so we will content ourselves with a few brief remarks. The description of $\Omega(L/K,2)$ involves both the Galois structure of O_L (i.e. additive Galois module structure) as well as the Galois structure of certain groups of local units (i.e. multiplicative Galois module structure). Chinburg showed that if L/K is tame, then $\Omega(L/K,2)=(O_L)$ in $Cl(\mathbb{Z}G)$. Hence, Taylor's proof of Fröhlich's conjecture implies that $\Omega(L/K,2)=W_{L/K}$. Chinburg conjectured that this equality in fact holds for all Galois extensions L/K. This is therefore a generalisation of Fröhlich's conjecture to the case of wild extensions (and in fact Snaith refers to it as the Fröhlich-Chinburg conjecture, although it is more commonly known as Chinburg's second conjecture). The invariant $\Omega(L/K,3)$ measures the $\mathbb{Z}G$-module structure of the group of S-units of L for any suitably chosen set of places S of L. It is closely connected with conjectures of Stark concerning the leading term of the Taylor expansion of the Artin L-function of L/K at $s=0$. Chinburg's third conjecture asserts that $\Omega(L/K,3)=W_{L/K}$. Finally, the invariant $\Omega(L/K,1)$ is the difference between $\Omega(L/K,2)$ and $\Omega(L/K,3)$, and Chinburg's first conjecture asserts that this invariant is always trivial.

The book under review originated as a graduate course given at The Fields Institute in 1993. It is aimed at advanced graduate students and other mathematicians who already have a thorough knowledge of basic algebraic number theory and who wish to become familiar with some of the techniques used in Galois module theory. The book is mainly concerned with additive Galois module theory and the second Chinburg conjecture (although the final chapter discusses the Galois structure of certain algebraic K-groups, and here the third Chinburg conjecture comes into play).

The first chapter of the book collects a certain amount of basic information that is needed for subsequent chapters. This includes material on Galois cohomology, local class field theory, the representation theory of finite groups, Artin and p-adic L-functions, and local Galois Gauss sums. The Explicit Brauer Induction homomorphism is described. This is a very useful technique that gives an explicit canonical formula for Brauer's induction theorem, and it leads to a number of new techniques which are described in the book (see also [Sn]).

Chapter 2 of the book is mainly concerned with the locally free classgroup $Cl(\mathbb{Z}G)$. Fröhlich's Hom-description of classgroups is explained, and the class $W_{L/K}$ is described. Chinburg's invariant $\Omega(L/K,2)$ is introduced, and Chinburg's second conjecture is proved for certain totally real abelian extensions of $\mathbb{Q}$ by using the theory of p-adic L-functions.

In Chapter 3, an account of the group-ring logarithm of M. Taylor and R. Oliver, together with some applications, is given. The description given here is different from most other accounts in that it is based upon the method of Explicit Brauer Induction. These methods also allow Snaith to improve upon certain previously known technical results concerning determinantal congruences. In the next chapter, the techniques developed in Chapter 3 are applied to give a proof of Fröhlich's conjecture for extensions L/K of p-power degree for any odd prime p.

The next two chapters of the book are concerned with Chinburg's second conjecture. Let Λ denote any maximal order in $\mathbb{Q}G$ that contains $\mathbb{Z}G$. Then there is a natural surjection $Cl(\mathbb{Z}G) \to Cl(\Lambda)$. The kernel subgroup $D(\mathbb{Z}G)$ of $Cl(\mathbb{Z}G)$ is defined to be the kernel of this surjection. It is independent of the choice of Λ, and it is usually non-trivial. Chapter 5 gives an account of a special case of a result of D. Holland ([H]). Holland's theorem says that Chinburg's second conjecture is true modulo $D(\mathbb{Z}G)$ for all extensions L/K with Galois group G. Snaith gives a different proof of this result (making use of Explicit Brauer Induction) assuming that all the wild decomposition groups of L/K are cyclic. I think that it would have been helpful to the reader if a proof of Theorem 5.3.1 had been included in this account. Chapter 6 is devoted to a very long calculation intended to verify Chinburg's second conjecture for abelian extensions $N/\mathbb{Q}$ whose Galois group is isomorphic to the quaternion group of order eight and which are not totally ramified above 2. This is a result of S. Kim ([Ki1], [Ki2]), but, as the author remarks, the approach taken here is somewhat different.

The final chapter is in some ways the most interesting, even though it is one of the shortest in the book. Here the author is concerned with the Galois structure of algebraic K-groups. By using work of B. Kahn ([K]) and by considering the relationship between $K_2(L)$ and $K_3(L)$, the author defines an invariant $\Omega_1(L/K, 3) \in Cl(\mathbb{Z}G)$. This is a higher analogue of Chinburg's third invariant $\Omega(L/K, 3)$, and the author conjectures that it should be related to the value of the Artin L-function at $s = -1$ in the same way that $\Omega(L/K, 3)$ is related to the leading coefficient of the Taylor expansion of the Artin L-function at $s = 0$. (The definition of this K-theoretic invariant was also obtained independently by G. Pappas.) He also explains how the Birch-Tate conjecture may be used to evaluate this new invariant in the class-group of the maximal order in a number of cases. He speculates that by considering the relationship between the K-groups $K_{2n}(L)$ and $K_{2n+1}(L)$ for $n > 2$, it ought to be possible to define higher invariants $\Omega_n(L/K, 3)$, and these invariants should be related to the conjectures of Lichtenbaum concerning the value of the Artin L-function at $s = -n$. The theory was just being worked out at the time the book was being written, and so only a somewhat tentative picture of the situation was available at that time. Our understanding has since improved; in particular for example, one now knows how to define Chinburg invariants attached to higher K-groups and, more generally, to certain motives. This circle of ideas is very much the subject of current investigations. We refer the reader to [BF1], [BF2], [CKPS1], and [CKPS2] for further details.

This book should be useful to mathematicians who are interested in learning about certain aspects of the theory of arithmetic Galois module structure, as well as to specialists in the area.

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