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*Automorphic forms on  $SL_2(\mathbf{R})$* , by A. Borel, Cambridge Tracts in Mathematics, vol. 130, Cambridge University Press, 1997, xiv+574 pp., \$47.95, ISBN 0-521-58049-8

*Automorphic forms and representations*, by D. Bump, Cambridge Studies in Advanced Mathematics, vol. 55, Cambridge University Press, 1997, xiv+574 pp., \$79.95, ISBN 0-521-55098-X

*Topics in classical automorphic forms*, by H. Iwaniec, Graduate Studies in Math., vol. 17, American Mathematical Society, Providence, RI, 1997, xii + 259 pp., \$39.00, ISBN 0-8218-0777-3

The study of automorphic forms involves a remarkable interplay between number theory, algebraic geometry, analysis and representation theory. Because of its multi-faceted nature, it is not surprising that each of the three introductory books under review approaches the subject from a different point of view.

Before turning to the books themselves, let us give a short overview of the subject. Recall that a lattice  $\Gamma \subset SL_2(\mathbf{R})$  is a discrete subgroup such that  $\Gamma \backslash SL_2(\mathbf{R})$  has finite invariant volume, and a classical modular form of weight  $k$  with respect to  $\Gamma$  is a holomorphic function  $f(z)$  on the Poincaré upper half-plane  $\mathcal{H}$  such that

$$(1) \quad f(\gamma(z)) = (cz + d)^k f(z) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Of course,  $\gamma(z)$  denotes the linear-fractional action of  $SL_2(\mathbf{R})$  on  $\mathcal{H}$ . The function  $f(z)$  must also satisfy a certain growth condition at the cusps of  $\Gamma$  which we omit. If  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$ , as in the case  $\Gamma = SL_2(\mathbf{Z})$ , then  $f(z)$  is invariant under  $z \rightarrow z + 1$  and  $f(z)$  can be expanded as a Fourier series:

$$f(z) = \sum_{n \geq 0} a_n e^{2\pi i n z}.$$

The growth condition insures that  $a_n = 0$  for  $n < 0$ . If  $a_0 = 0$ ,  $f$  is called a cusp form.

The rich interaction of themes is already apparent in the simplest modular forms, the Eisenstein series on congruence subgroups  $\Gamma$  of  $SL_2(\mathbf{Z})$ . For  $\Gamma = SL_2(\mathbf{Z})$ , the Eisenstein series of weight  $2k$  for  $k > 1$  is defined by the well-known formula

$$G_{2k}(z) = \sum_{\substack{(c,d) \in \mathbf{Z}^2 \\ (c,d) \neq (0,0)}} (cz + d)^{-2k}.$$

It is immediate that the series converges to a holomorphic function and satisfies the transformation law (1) for weight  $2k$ . The Fourier expansion of  $G_{2k}(z)$  suggests a connection with number theory:

$$G_{2k}(z) = 2\zeta(2k) + c_{2k} \sum_{n \geq 1} \sigma_{2k-1}(n) e^{2\pi i n z}.$$

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Here  $c_\ell = 2(2\pi i)^\ell / \ell!$ ,  $\sigma_\ell(n) = \sum_{d|n} d^\ell$ , and  $\zeta(s)$  is the Riemann zeta-function ([Ser]). In fact, the connection is deeper than appears at first sight. According to the Siegel-Weil formula, the Fourier coefficients of Eisenstein series are closely connected to the number of ways of representing an integer  $n$  by quadratic forms. For example, Jacobi's formula for the number  $r_4(n)$  of ways of writing  $n$  as a sum of four squares:

$$r_4(n) = 8(2 + (-1)^n) \sum_{\substack{d|n \\ d \text{ odd}}} d$$

is equivalent to the assertion that  $r_4(n)$  is equal to the  $n^{\text{th}}$ -Fourier coefficient of a certain Eisenstein series of weight 2 on a congruence subgroup of  $SL_2(\mathbf{Z})$ . The constant term in the Fourier expansion of an Eisenstein series is arithmetically significant in a different way. It is a special value of the Riemann zeta-function (or a Dirichlet  $L$ -function in the case of congruence subgroups), and in the work of Ribet [Ri] and of Mazur and Wiles this fact is the basis of an important link between the  $p$ -adic properties of these special values and the theory of cyclotomic fields. The values of Eisenstein series at points  $z_0 \in \mathcal{H}$  such that  $\mathbf{Q}(z_0)$  is a quadratic imaginary field give a third link to number theory. They are essentially the special values of Hecke  $L$ -functions for the quadratic field ([W2]). This is an example of the many relations, both known and conjectural, that exist between automorphic forms and special values of  $L$ -functions ([Ra]).

There is a more general definition of Eisenstein series on  $SL_2(\mathbf{R})$  that includes the classical Eisenstein series  $G_{2k}(z)$  as a special case ([Bo], §10.8). This general definition also includes the spectral Eisenstein series of weight zero and parameter  $\lambda \in \mathbf{C}$ :

$$E(z, \lambda) = \operatorname{Im}(z)^{\frac{\lambda+1}{2}} \sum_{\substack{(c,d) \in \mathbf{Z}^2 \\ (c,d) \neq (0,0)}} |cz + d|^{-\lambda-1}.$$

This series converges absolutely if  $\operatorname{Re}(\lambda) > 1$ . The function  $E(z, \lambda)$  is not a classical modular form since it is not holomorphic, but it is a real analytic eigenfunction of the Laplacian  $\Delta$  on  $\Gamma \backslash \mathcal{H}$  with eigenvalue  $\frac{1}{4}(1 - \lambda^2)$ . As such, it is an example of a Maass form. These Eisenstein series and their variants for congruence subgroups are the basic eigenfunctions in the continuous part of the spectral decomposition of the Laplacian  $\Delta$  on  $\mathcal{H}$ . Thus they are upper half-plane analogues of the exponential functions  $e^{\lambda t}$ . However, just as standard Fourier analysis is based on the exponentials  $e^{\lambda t}$  for  $\lambda \in i\mathbf{R}$ , the spectral analysis of the self-adjoint operator  $\Delta$  needs to be carried out using the Eisenstein series  $E(z, \lambda)$  with  $\lambda \in i\mathbf{R}$ , where the series no longer converges. One of the basic results due to Selberg asserts that  $E(z, \lambda)$  has a meromorphic continuation in  $\lambda$  to the entire complex plane with no poles on the imaginary axis. This is the starting point of the spectral theory of automorphic forms. On the other hand, the spectral Eisenstein series also appear as kernel functions in the so-called Rankin-Selberg convolution of two cusp forms  $f(z)$  and  $g(z)$  of weight  $k$  ([Bu], §1.6; [I1], §13):

$$L(s, f, g) = \int_{\Gamma \backslash \mathcal{H}} f(z) \overline{g(z)} \operatorname{Im}(z)^{k-2} E(z, s) dz.$$

Up to a simple factor involving the gamma function,  $L(s, f, g)$  is equal to the Dirichlet series  $\sum a_n b_n n^{-s}$  where  $a_n$  and  $b_n$  are the Fourier coefficients of  $f$  and  $g$ .

It plays an important role in a wide range of problems, some of which are mentioned below. In any case, the Eisenstein series provide a first illustration of the rich set of links between automorphic forms, number theory and analysis.

To make the connection with representation theory, it is necessary to define automorphic forms as functions  $\varphi$  on  $\Gamma \backslash SL_2(\mathbf{R})$  rather than on the symmetric space  $\mathcal{H}$ . More generally, one defines automorphic forms on  $\Gamma \backslash G(\mathbf{R})$  where  $G(\mathbf{R})$  is the group of real points of a reductive algebraic group  $G$  over  $\mathbf{Q}$  and  $\Gamma \subset G(\mathbf{R})$  is a lattice. The group  $G(\mathbf{R})$  acts by right translation (denoted  $\rho$ ) on various spaces of functions on  $\Gamma \backslash G(\mathbf{R})$ . For example, we obtain a unitary representation of  $G(\mathbf{R})$  on  $L^2(\Gamma \backslash G(\mathbf{R}))$ . A smooth function  $\varphi$  on  $\Gamma \backslash G(\mathbf{R})$  is called an automorphic form if it satisfies a growth condition and two finiteness conditions. The first finiteness condition requires that  $\varphi$  transform by a finite-dimensional representation under a fixed maximal compact subgroup of  $G(\mathbf{R})$ . The second requires that the space  $\{\rho(z)\varphi\}$  be finite-dimensional, where  $z$  ranges over the center of the enveloping algebra of  $G(\mathbf{R})$  and  $\rho(z)$  is the derived action. These conditions imply that  $\varphi$  is real-analytic. In the case  $G(\mathbf{R}) = SL_2(\mathbf{R})$ , to go from a classical modular form  $f(z)$  of weight  $k$  to an automorphic form  $\varphi_f$  on  $\Gamma \backslash SL_2(\mathbf{R})$ , one defines  $\varphi_f(g) = (ci + d)^{-k} f(g(i))$  where  $(c, d)$  is the bottom row of  $g$  ([Bo], §5.13, [Bu], §3.2, or [G1]). If  $f$  is cuspidal, then  $\varphi_f$  is square-integrable and generates an irreducible subrepresentation  $V_f$  of  $L^2(\Gamma \backslash SL_2(\mathbf{R}))$  which is isomorphic to the discrete series representation  $D_{k-1}$ . This leads to an isomorphism ([GGPS], [De])

$$(2) \quad S_k(\Gamma) \xrightarrow{\sim} \text{Hom}_{SL_2(\mathbf{R})}(D_{k-1}, L^2(\Gamma \backslash SL_2(\mathbf{R}))),$$

where  $S_k(\Gamma)$  is the space of all cusp forms of weight  $k$  with respect to  $\Gamma$ . A similar isomorphism holds if we replace  $S_k(\Gamma)$  by the space of square-integrable Maass forms of fixed eigenvalue  $\frac{1}{4}(1 - \lambda^2)$  and  $D_{k-1}$  by the corresponding principal series representation  $\pi_\lambda$ .

The theory of automorphic forms has developed in many directions throughout this century as new themes and generalizations have progressively been added. Siegel developed the theory of modular forms on the symplectic group and applied it to prove the so-called Siegel-Weil formula mentioned above. This is a vast generalization of Jacobi's result to the problem of counting integral solutions  $X$  to the equation  ${}^t X Q X = R$  where  $Q$  and  $R$  are integral symmetric matrices (quadratic forms) of size  $m$  and  $n$  and  $X$  is an  $m \times n$  integral matrix. Although Siegel's papers have a strongly analytic appearance, Weil [W1] showed in the 1960's that Siegel's results could be reformulated as a uniqueness assertion for a certain distribution on the space of the so-called oscillator representation of the two-fold cover of the symplectic group. This gave a representation-theoretic framework for the study of theta-series and was the starting point of Roger Howe's theory of dual reductive pairs in the 1970's, which has since become a basic component of the theory of automorphic forms ([Ho], [Pr], [Wa]). The Siegel-Weil formula itself has been significantly extended in the work of Kudla and Rallis.

In a different direction, Hecke initiated the study of L-series attached to modular forms. The Hecke L-series attached to a classical cusp form  $f(z) = \sum_{n \geq 1} a_n e^{2\pi i n z}$  on  $SL_2(\mathbf{Z})$  is the Dirichlet series  $L(s, f) = \sum_{n \geq 1} a_n n^{-s}$ . Hecke proved that  $L(s, f)$  has an analytic continuation and satisfies a functional equation. Building on Mordell's study of the Ramanujan  $\Delta$ -function, Hecke defined a commutative ring of operators on the space of modular forms, the ring of Hecke operators. The basic fact is that if  $f$  is an eigenfunction of the Hecke operators, then  $a_1 \neq 0$  and with

the normalization  $a_1 = 1$ , the L-series has an Euler product of the form

$$L(s, f) = \prod_p (1 - a_p p^{-s} + p^{k-1-2s})$$

([Bu], §1.4; [I1], §6-7). Of course, all of this extends to congruence subgroups of  $SL_2(\mathbf{Z})$  and also to  $GL(n)$  and other reductive groups ([J2],[GS]).

Several new lines of investigation were developed in the 1950's: the spectral theory of automorphic forms due to Maass-Roelcke-Selberg, the representation-theoretic approach of Gelfand and Harish-Chandra, and the arithmetic theory of Eichler and Shimura relating the zeta-functions of modular curves and modular forms. Eichler-Shimura theory ([Sh]) gave rise in the 1960's to the study of  $\ell$ -adic representations of  $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$  attached to modular forms pioneered by Serre. Closely related to this was the development of the theory of  $p$ -adic modular forms and eventually, Hida's  $p$ -adic deformation theory of modular forms [Hi]. All of these are key ingredients in Wiles' proof of Fermat's Last Theorem. At the same time, Shimura developed his general theory of canonical models for Shimura varieties which, among other things, opened up the possibility of generalizing Eichler-Shimura theory to higher rank groups (see [BR] for a survey, [LR] for a detailed realization of this program for the group  $U(3)$ , and [La] for the case of  $GL(n)$  over a function field which includes a lot of foundational material).

The development which most unified the subject, clarifying and to a large degree redefining its goals, was the formulation by Langlands in the 1960's of the principle of functoriality [L1]. It implies the existence of an intricate web of relations which are known in important special cases but otherwise remain conjectural. One of the most dramatic aspects of these conjectures from the historical point of view is that they include a non-abelian generalization of the Artin reciprocity law. Artin's law is a vast generalization of the law of quadratic reciprocity, and it is the foundational result of abelian class field theory, so-called because it describes the abelian extensions of a number field in terms of generalized ideal classes. An extension of the reciprocity law to include non-abelian extensions had been sought by number theorists, but it could not have been found without Langlands' insight that such a law must be formulated as a statement about automorphic forms, phrased in the language of infinite-dimensional representation theory ([Ro], [G2]). In other words, the non-abelian generalization of a Dirichlet character turned out to be an automorphic form, and with this realization, the elaborate machinery of harmonic analysis on reductive groups as pioneered by Harish-Chandra was incorporated into number theory.

The principle of functoriality also placed non-abelian reciprocity in the much larger context of “functorial relations” between automorphic representations on different groups. The L-group construction is needed to describe these relations, so we refer to [G2], [K1], or [Bu],§3.9 for a general overview. However, a typical example is the base change lifting, which gives a correspondence between automorphic forms on  $GL(2)$  over a field  $F$  and automorphic forms on  $GL(2)$  over an extension  $E$  of  $F$ . This lifting is a non-abelian generalization of the norm map from  $E$  to  $F$ . Its existence for  $GL(2)$  and, more generally, for  $GL(n)$  is known for extensions  $E/F$  with solvable Galois group ([AC],[L3]). The power of functoriality was illustrated in [L3], where base change for  $GL(2)$  was used to prove Artin's conjecture for most complex two-dimensional Galois representations with solvable image. The qualifier “most” was later removed by Tunnell, and the resulting theorem was another

key ingredient in Wiles' work (see [Ro] for an exposition). The main tool in this work is the trace formula, although the Rankin-Selberg integral ([JS]) also plays a critical role. The broad point is that functoriality shifts the focus away from the classical activity of studying individual automorphic forms and directs it instead at the problem of proving functorial relations between spaces of automorphic forms on different groups. In the other direction, functoriality has had a major impact on representation theory itself. Basic problems such as the local Langlands conjecture ([BuK], [Ku], [Mo]), the fundamental lemma ([Ko], [LS]), and Arthur's conjectures [A2] have provided the impetus for a great deal of research in representation theory over the past two decades. Anyone interested in pursuing these topics will find the four volumes of Proceedings [BM], [BC], [CM], and [BK] indispensable. See [L3] and [L4] for some interesting historical reflections and speculations.

We now turn to the books under review. *Automorphic forms on  $SL_2(\mathbf{R})$*  by Armand Borel has the sharpest focus of the three. The author's goal is to provide a complete and accessible exposition of Selberg's spectral theory of automorphic forms on  $SL_2(\mathbf{R})$ . He adopted the setting of representation theory rather than that of analysis on the symmetric space as in Selberg's papers, and therefore the main object of study is the right regular representation  $\rho$  of  $SL_2(\mathbf{R})$  on  $L^2(\Gamma \backslash SL_2(\mathbf{R}))$  for an arbitrary lattice  $\Gamma$ . One begins by singling out the invariant subspace of cusp forms. To define it, let  $N$  be an arbitrary conjugate of the subgroup of unipotent matrices of the form

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

and let us say that  $N$  is cuspidal if  $(N \cap \Gamma) \backslash N$  has finite invariant volume. The constant term of an automorphic form  $\varphi$  along  $N$  is the function

$$\varphi_N(g) = \int_{(N \cap \Gamma) \backslash N} \varphi(ng) dn$$

on  $N \backslash SL_2(\mathbf{R})$ . We say that  $\varphi$  is cuspidal if  $\varphi_N = 0$  for all cuspidal  $N$ . If  $N$  is cuspidal, then  $(N \cap \Gamma) \backslash N \approx \mathbf{Z} \backslash \mathbf{R}$ , and if  $\varphi = \varphi_f$  as above, then  $\varphi_N$  may be identified with the constant term in a Fourier expansion of the associated classical modular form  $f$ . The subspace  $L_0^2$  of all cusp forms is invariant under  $\rho$ , and we have a decomposition

$$L^2(\Gamma \backslash SL_2(\mathbf{R})) = L_0^2 \bigoplus L_e^2$$

where  $L_e^2$  is the orthogonal complement of  $L_0^2$ .

The book provides an explicit description of  $L_e^2$  in terms of Eisenstein series. We may decompose  $L_e^2$  as  $L_e^2 = L_d^2 \bigoplus L_c^2$  where  $L_d^2$  is the sum of all irreducible invariant subspaces of  $L_e^2$  and  $L_c^2$  is its orthogonal complement. Thus  $L_d^2$  is the part of  $L_e^2$  that decomposes discretely as a direct sum of irreducible representations. The two main results are

- (1)  $L_d^2$  is generated by the residues of Eisenstein series for parameters  $\lambda$  in  $(0, 1]$  at which a pole occurs (Theorem 16.6), and
- (2)  $L_c^2$  is isomorphic to a sum of continuous direct integrals of principal series representations of  $SL_2(\mathbf{R})$  (Theorem 17.7).

The space  $L_d^2$  always contains the space of constant functions corresponding to a residue at  $\lambda = 1$ . If  $\Gamma$  is a congruence subgroup, there are no other residues and in this case  $L_d^2$  is one-dimensional. The isomorphism in (2) is given explicitly in

terms of the spectral type of Eisenstein series  $E(z, \lambda)$  defined above with unitary parameter, i.e.,  $\lambda \in i\mathbf{R}$ .

As noted above, to speak of  $E(z, \lambda)$  for unitary  $\lambda$  it is necessary to meromorphically continue the Eisenstein series. At the same time, one also proves a functional equation relating  $E(z, \lambda)$  and  $E(z, -\lambda)$ . There are many ways of doing this for  $SL(2)$  or  $GL(2)$ . Bump ([Bu], §3.7) and Iwaniec ([I1], §13.3) exhibit the meromorphic continuation for Eisenstein series on congruence subgroups of  $SL_2(\mathbf{R})$  by computing their Fourier expansions and observing that the coefficients can be continued. Borel presents a beautiful proof, attributed to J. Bernstein and Selberg independently, which deduces the meromorphic continuation in a very direct way from the corresponding property of the resolvent of a compact operator. This proof works for all lattices  $\Gamma$  in  $SL_2(\mathbf{R})$  and is a simplification of Selberg's original proof. A similar proof for  $GL(2)$  over the adeles is given in [J1]. The general theory of Eisenstein series on reductive groups due to Langlands has been given an excellent exposition in [MW].

In essence, the above results say that  $L_e^2$  has a relatively simple structure. However, the truth is that we are ultimately much more interested in  $L_0^2$ , which is the mysterious and arithmetically significant part of  $L^2(\Gamma \backslash SL_2(\mathbf{R}))$ . The importance of understanding  $L_0^2$  is that it is a prerequisite to the study  $L_0^2$ . Indeed, the isomorphism of (2) is one of the main ingredients in the derivation of the Selberg trace formula, which gives an expression for the trace of certain integral operators acting on  $L_0^2$ . The trace formula is a powerful tool in the study of automorphic forms on congruence subgroups of  $SL_2(\mathbf{R})$  as well as groups of higher rank. By contrast, very little is known about  $L_0^2$  for general non-arithmetic subgroups, although there are some interesting conjectures ([S2]).

The main results just described are contained in the second half of the book. The first half develops all of the necessary background starting nearly from scratch. The exposition is enhanced by helpful comments, explanations, and references. Above all, the author has followed an approach that will facilitate the reader's access to the general theory as covered in the basic reference [MW]. As Borel himself observes in the chapter entitled "Concluding Remarks", the endpoint of his book is really the starting point of the theory. The natural next step would be to develop the trace formula and its applications. However, an excursion into the trace formula would have altered the length and balance of the book, and fortunately there are several good expositions of it for  $GL(2)$  available ([A4], [DL], [G1], [G3], [He], [I2], [K2]). As it is, this is a beautiful and masterfully written volume. It will be of great value to anyone seeking a pathway into the spectral side of automorphic forms.

To introduce *Automorphic forms and representations* by Daniel Bump, recall that the principle of functoriality highlights the connection between class field theory and automorphic forms. Just as class field theory is divided into a local and global part, the theory of automorphic forms, when formulated in terms of adelic reductive groups over global fields, has as its local counterpart harmonic analysis on reductive groups over local fields. This local/global framework for  $GL(2)$  over an arbitrary global field was developed in volume 114 of the Springer Lecture Notes known simply as "Jacquet-Langlands" [JL]. In Jacquet-Langlands, the starting point is the notion of an automorphic form on the quotient  $GL_2(F) \backslash GL_2(\mathbf{A}_F)$ , where  $F$  is an arbitrary global field and  $\mathbf{A}_F$  is its adele ring. Recall that  $\mathbf{A}_F$  is the subring of sequences  $(a_v)$  in the direct product  $\prod_v F_v$ , where  $v$  ranges over all places of  $F$  (including the infinite ones if  $F$  is a number field), such that  $a_v$  is a

$v$ -adic integer for almost all  $v$ . There is an adele topology which makes  $\mathbf{A}_F$  into a topological ring and hence  $GL_2(\mathbf{A}_F)$  into a topological group. Under this topology,  $GL_2(F)$  is a discrete subgroup of  $GL_2(\mathbf{A}_F)$ . Considering adelic automorphic forms is essentially equivalent to considering automorphic forms with respect to all congruence subgroups  $\Gamma$  without having to specify  $\Gamma$  in advance. In the case  $GL_2(\mathbf{Q})$ , it is equivalent to considering automorphic forms on all quotients  $\Gamma \backslash SL_2(\mathbf{R})$  for  $\Gamma$  a congruence subgroup. Although the adele setting applies only to congruence subgroups, it has two great advantages over the classical setting: it makes manifest an underlying product structure over the places of  $F$ , and it allows us to treat all global fields  $F$  in a uniform way. For example, the theory makes no distinction between classical modular forms and Hilbert modular forms.

In the adelic approach, the map (2) is refined to a map  $f \rightarrow \pi(f)$  associating an infinite-dimensional irreducible unitary representation  $\pi(f)$  of  $GL_2(\mathbf{A}_{\mathbf{Q}})$  to each classical cusp form  $f$  which is an eigenfunction of the Hecke operators. The representation  $\pi(f)$  occurs as a constituent of the space  $L_0^2$  of cusp forms on  $GL_2(\mathbf{Q}) \backslash GL_2(\mathbf{A}_{\mathbf{Q}})$ . Each irreducible constituent  $\pi$  of  $L_0^2$  is isomorphic, as an abstract representation, to a “restricted” tensor product  $\otimes' \pi_v$ , where  $v$  ranges over all places of  $\mathbf{Q}$  and  $\pi_v$  is an irreducible unitary representation of  $GL_2(\mathbf{Q}_v)$  ([Bu], §3.4; [De]). This factorization is the representation-theoretic source of the Euler product decomposition of the Hecke L-functions attached to  $f$  (or to an arbitrary cuspidal representation). It also enables us to clearly divide the work into two parts: a local study of the individual “abstract” representations  $\pi_v$ , and a study of the global representation  $\pi$ , regarded as a “physical” subspace of  $L_0^2$ .

The book by Bump is intended to provide a motivated and friendly access to Jacquet-Langlands theory for  $GL(2)$  and the Rankin-Selberg method. It covers a broader terrain than [Bo], but is less tightly organized and considerably longer (574 pages). The first chapter treats modular forms and Rankin-Selberg from a classical (non-adelic) point of view. As an application of the Rankin-Selberg method, the Doi-Naganuma approach to the base change lifting for quadratic extensions is presented. This provides a nice example of the way in which L-function techniques are used to prove results about functoriality. The remaining three chapters treat the local and global theory for  $GL(2)$  (exclusive of the trace formula) together with a discussion of Rankin-Selberg in the adelic framework. Apart from the meromorphic continuation and some background technical results, there is not much overlap with [Bo].

Bump’s book contains a lot of interesting information, motivational explanations, and good exercises. Since many of the arguments presented are computational, the author is careful to explain how the particular computations fit into the broader picture. In some cases, proofs or certain details of a proof are omitted, but the author provides references, and he encourages the reader to consult them to gain a fuller understanding. Another feature of the book is its conversational tone. While the informality is for the most part welcome, I occasionally found it distracting. For example, in the middle of a technical section (3.5) dealing with Whittaker models and the Multiplicity One Theorem, there are inserted a couple of pages of informal comments on a wide range of topics, including Tate’s thesis, Satake parameters, tempered principal series, good reduction of modular curves, Eichler-Shimura theory, Maass forms, and best bounds on the Ramanujan conjecture. After this interlude, the author abruptly resumes his technical discussion, which one eventually realizes is leading towards a proof of the local and global functional

equations. The book's organizational problem is exacerbated by the arrangement of the chapters, which places Chapter 4, dealing with the local theory of  $GL(2)$  over a  $p$ -adic field, after Chapter 3, dealing with global automorphic representations of  $GL_2(\mathbf{A}_F)$ . The author even admits in the introduction that this is not the logical order of the material. But this leads to numerous forward references to Chapter 4 in Chapter 3.

Despite these shortcomings, I would recommend [Bu] to graduate students or anyone else who wants to get into the subject. There is a great deal of foundational material here. Any effort put into reading it and working exercises will be rewarded with a good understanding of the basics.

To place the third book in context, we remark that over the past three decades, research in the Langlands program has been pursued along three main lines: via L-functions ([GS], [JS]), via dual reductive pairs (theta liftings) ([Ho], [Pr], [Wa]), and via the trace formula ([A1], [A3]). There are numerous points of intersection between these viewpoints, and they are all aimed at understanding as much as possible about the functoriality conjecture mentioned above. By contrast, one can take the point of view that automorphic forms are primarily of interest because of the concrete analytic information they give us about classical problems. In this optic, functoriality is a tool rather than an end in itself, and a wide range of other methods from analytic number theory play an equally important role.

This is the approach of Iwaniec in *Topics in classical automorphic forms*. Like the other two books under review, Iwaniec devotes several chapters to standard background material: the modular group, Eisenstein series, Hecke operators, L-functions, etc. However, the main focus is on two problems: (1) estimating the size of the Fourier coefficients of a modular form and (2) representing integers by quadratic forms.

Iwaniec approaches (1) through Poincaré series and Kloosterman sums in Chapters 4 and 5. The Ramanujan-Petersson (RP) conjecture asserts that the  $n^{\text{th}}$  Fourier coefficient  $a_n$  of a classical cusp form  $f$  on a congruence subgroup of  $SL_2(\mathbf{Z})$  satisfies  $|a_n| = O(n^{\frac{k-1}{2} + \varepsilon})$ . As is well-known, Deligne reduced the RP conjecture to the Riemann hypothesis over finite fields, which he later proved. However, Deligne's result is only part of the story. The RP conjecture can also be stated for Maass forms and for holomorphic forms of half-integral weight. The RP conjecture for classical forms or Maass forms would follow from a part of the functoriality conjectures, as observed in [L1]. In fact, the functorial approach applies in principle to cuspidal representations  $GL(n)$  for all  $n$ , although it still seems far from realization. Alternatively, one can try to prove the RP conjecture using analytic techniques. This is especially crucial in the half-integral weight case since functoriality does not apply, even conjecturally. Iwaniec did the pioneering work on the half-integral weight RP conjecture ([I3]). He presents some of his results in this book. They are based on delicate manipulations of estimates for Kloosterman sums and lead to non-trivial estimates of Fourier coefficients in both the integral and half-integral weight case. See [LRS], [S3] for recent progress on the archimedean analogue of the RP conjecture.

To treat problem (2), Iwaniec devotes Chapters 9 and 10 to the basic theory of theta functions on the upper half-plane attached to a positive definite quadratic form  $Q$ . They are of interest for their Fourier coefficients, the  $n^{\text{th}}$  of which is equal to the number  $r(n, Q)$  of integral solutions to  $Q(x) = n$ . In Chapter 11,

the Hardy-Littlewood circle method is used to obtain an estimate for  $r(n, Q)$ . The circle method is an alternate to the representation-theoretic approach to  $r(n, Q)$  via the Siegel-Weil formula. Actually, the Siegel-Weil formula only gives information about a weighted average of the numbers  $r(n, Q)$  for  $Q$  in a genus class, so for some problems the circle method leads to stronger analytic results. As an application of the circle method and the estimates on Fourier coefficients, Iwaniec proves that the integral solutions are asymptotically equidistributed over the ellipsoid  $Q(x) = n$  (§11.6). See [Du] for an overview of recent progress and open problems in this area.

Iwaniec treats a number of other topics, though sometimes without detailed proofs: newforms, Weil's converse theorem, automorphic forms attached to Hecke characters and elliptic curves, Eisenstein series, and the Rankin-Selberg method. Although analytically demanding in parts, the exposition is clear, and helpful explanatory comments are included throughout. If I have one small complaint, it is that the author did not include an overview section similar to the “Concluding Remarks” section in [Bo]. Such a section could have described the state of the field and provided the reader with a useful guide of where to go next. Nevertheless, this is an excellent place to begin the study of the analytic approach to modular forms. The book [S1] is a good companion to [I1], dealing with similar topics in a complementary manner and going into applications to other fields such as graph theory.

Anyone encountering automorphic forms for the first time should begin by reading the last chapter of Serre's beautiful book *A course in arithmetic* [Ser]. Where to turn next will depend on personal tastes, but fortunately, there now exist several good choices of books and survey articles. Each of the books under review is a welcome addition to this growing expository literature.

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