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The topological classification of stratified spaces, by Shmuel Weinberger, Chicago Lectures in Math., Univ. of Chicago Press, Chicago, IL, 1994, xiv+283 pp., $47.50, ISBN 0-226-88566-6

For several decades, topologists have had a scheme, called surgery theory, for classifying high dimensional manifolds. This scheme reduces this classification problem to a mix of homotopy theory, characteristic classes (much like the Pontrjagin classes of the tangent bundle, but somewhat more refined), and algebra (more precisely, some algebraic K-theory) involving the fundamental group. (Of course, these ingredients are not made of entirely different elements. Thus, while there were very valuable initial successes in dealing with the associated algebra using largely algebraic number theory or algebraic constructions, in recent years most of the advances in high dimensional manifold theory have been achieved by thinking about the fundamental group geometrically, or by thinking of it using a mix of algebra or global analysis with geometry.) We do not here discuss the development of this powerful general scheme of classification for manifolds; for some discussions, see the reviews [R], [We].

However, problems from very many areas of mathematics naturally lead to geometrical questions about spaces much more general than manifolds. These are spaces which, because they contain singularities, lack the local homogeneity of manifolds. In a sense, this phenomona already arises in the special case of manifolds with boundary; for these Wall gave early on a quite adequate extension of the classification scheme (although there the algebraic aspects became a bit awkward). However, there are natural and far more variegated and complicated situations where one still wishes to accomplish classification up to homeomorphism, e.g., algebraic varieties, quotients of group actions, compactifications of Riemannian manifolds with some "geometry at infinity" being encoded in the compactification, etc.

For these problems of more general spaces, classical surgery theory is inadequate. The goal of the book under review is to introduce and present a very general scheme for these problems and also to "work out" some important or interesting examples of such a theory.

Needless to say, many of these special cases have previously been the topic of intensive and extensive efforts by many mathematicians (and, indeed, not just topologists). It is remarkable that there is a general theory that sheds light on such diverse subjects. The ideas that feed into this perspective are a synthesis and extension of ideas which arose first in one or the other of the natural examples and fields of applications, and in some cases arose independently in several areas. (Indeed, one might argue that a sufficiently perceptive analysis of manifolds with boundary and of manifolds with submanifolds could have suggested much of the general theory!) So, the history is complicated and interwoven with many other research problems, as is any description of the motivations or background, etc. Unfortunately, in the space of a review, it will not be possible to get to most of it.

A stratified space is a space which has pieces (open strata) that are manifolds which are put together in "not too wild" a fashion. There are various notions in

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the literature of what these words between quotation marks mean. For studying the homeomorphism problem, and for many other purposes, a relative latecomer introduced by Quinn [Q1] is particularly convenient. In this set-up one only assumes local homotopical conditions, but it turns out that these suffice to yield very useful geometrical properties, such as homogeneity in the interior of strata, isotopy extension, etc. (See [Q1], [Hu].) This structure is satisfactorily minimal, so that given a space with a filtration, if it has a stratified structure in this sense, it is unique. (This can be contrasted with such more delicate topological structures as a piecewise-linear or a smooth structure on a topological manifold; these, when they exists are rarely unique. This kind of difficulty afflicts Whitney stratifications as well: while they often exist, they are not preserved by homeomorphism. Indeed, not enough attention has been paid in the literature to the question of what is the structure group of the bundles that arise in a Whitney stratification.)

To get a feeling for what kind of spaces are allowed by this stratification notion, consider embeddings and group actions. A submanifold of a manifold will give rise to a 2-stratum space if the submanifold is locally flat; i.e., locally at each point on the submanifold one can describe a neighborhood which looks like a vector space with a vector subspace. Notice that one doesn't assume anything about how overlapping charts fit together. It's even more striking that in the case of a finite group action one need only suppose that the fixed point set of each subgroup is locally flat. The fact that there is then some common local model and that the action looks the same near any two fixed points (in the same component of the fixed point set) is a remarkable consequence.

So such stratified spaces arise naturally whenever one is interested in topological homeomorphism problems, and are convenient even when the initial objects have some richer structure, e.g., piecewise linearity or "smoothness" in any sense.

One quite early and natural problem of that sort which led to a great deal of the development of the foundations for this work, and which remains of interest, is that of deciding whether two different representations of a finite group on a vector space are topologically conjugate. Classical work of DeRham had shown that nothing surprising could occur for piecewise-linear homeomorphisms, and it was conjectured that in general there could be no such homeomorphisms of different representations. As evidence, Schultz [Sc] and Sullivan (unpublished) showed that this was correct for odd order p-groups. However, about twenty years ago Shaneson and I gave counterexamples by showing that for cyclic groups of even order there were such "nonlinear similarities" of periodic matices. (See [CS1].) This meant that the inductive methods invoked by DeRham, which used around the strata rigid local structures analogous to the tubular neighborhoods of differential topology, had no chance of direct repair. In some sense, it is this absence of rigid local structure that enables Weinberger to prove his powerful classification theorems.

Soon after these examples, Hsiang and Pardon [HP] on the one hand and Madsen and Rothenberg [MR] on the other showed that the original DeRham conjecture was true for odd order groups. The former proof was a rather direct attack at the conjecture (and proved it in some other cases as well); the latter was a systematic exploration of the terrain and led to very substantial classification results for odd order group actions. The approach that Madsen and Rothenberg took was a long (several hundred pages) and complicated induction that established an equivariant transversality theorem in the setting of group actions. However, for even order groups (even for Z/2) transversality fails, and therefore so does their method, in accord with the counterexamples above. Nonetheless, many of the ideas used in the course of their proofs, many of which they had borrowed from Chapman, Ferry, Quinn, Anderson and Hsiang and others (see, e.g. [ChF], [Q2I;II], [AH], [FP], [ACFP]), were also to be used by Weinberger. A few years later, using tools from Lipschitz analysis, Rothenberg and Weinberger (see [RoW]) extended the characteristic classes of [MR] to arbitrary compact groups, although their topological significance was not at all clear. I conjectured (based on joint work with Weinberger on semifree group actions [CW1]) that under suitable fundamental group and dimension restrictions, these invariants would detect the conjugacy class of finite group actions on manifolds. One of the results in chapter 13 of this book affirms that conjecture.

Another important characteristic class whose deep significance is illuminated by this work is the L-class for spaces with even codimensional singularities, introduced using the intersection homology theory in [GMI;II]. If one assumes that the links of all strata are simply connected, then it turns out, see [CW2], that a family of these classes plays exactly the same role in topological classification of stratified spaces that ordinary L-classes (that is, the L-polynomials in the Pontrjagin classes) do in the classification of manifolds. From the point of view of the general theory discussed here, this result is now obtained as a calculation obtained from a comparison of the general classification scheme with intersection homology theory. (In more detail, a priori maps to a sum of K-homology groups can be obtained by viewing the intersection homology complexes of each stratum [GMII] as self-dual sheaves in the sense of [CS2], [CSW]); the maps induce a quasi-isomorphism between the L-cosheaf and the sum of K-homology groups, yielding the result.)

The above examples are typical. In order to actually unravel and calculate what's going on in particular cases of interest (e.g., to classify spaces in a given stratified homotopy type) it is most likely that one will have to compare the natural invariants of the given setting, such as these L-classes, in order to interpret the general theory. One does have, for any stratified space, a very general characteristic class which simultaneously generalizes all the examples described above – however, it lies in quite an abstract and, at first glance, daunting place ("an L-cosheaf homology group") which inevitably requires investigation to be useable in practice. In some sense, as in classical surgery, while an enormous amount of information is compressed into a few exact sequences, it still is the case that many examples and applications will require new geometric insights or algebraic constructions. Nevertheless, this machinery will instantly process any such information efficiently and usefully and will very significantly expand its range of applications.

Let me now turn to the book under review. Part I is devoted to a quick "review" of high dimensional manifold theory, updating the classical treatment from [Br], [Wa] and giving a more geometrically oriented version of what can be found in [Ra]. The final chapter of this part revisits many classical examples, sometimes giving interesting new proofs, and also highlighting the rich connections between the geometrical topology and elliptic operators. (This theme recurs in this book, which is appropriate as it is a powerful current in recent work – as the two volumes [FRR] attest.)

Part II develops the theory of stratified spaces. It reviews an early, often overlooked, paper of Browder and Quinn [BQ] that was an early attempt at a classification theory which had built in many strong transversality assumptions; as such, it did not support a great many applications. Nevertheless, what has become a key dramatis personum, the Browder-Quinn L-group (a large-scale geometrical generalization of the Wall surgery obstruction groups) arose already in this setting; the spectral cosheaves that enter in Weinberger's general theory have as local stalk homotopy groups these groups. This part also contains an exposition of "controlled topology" and some of its main applications; this will be of independent interest to some researchers, apart from the general theory of stratified spaces. From the point of view of the central thrust of this book, the controlled topology enters in the proofs of the kind of stratified h-cobordism theorem (due to [Q1] , modulo an issue of realization that was established in the papers [HTWW], [Hu]) and in the general classification theorem.

Part III gives a wealth of applications directly to classification problems (including the ones mentioned above), as well as to directions that at first glance appear quite unrelated to these, from rationality theorems for eta invariants, to proofs of some cases of the famous Novikov conjecture and counterexamples to the equivariant topological rigidity conjecture.

This book is in the first place a research monograph presenting remarkable and powerful results, but it is generously expanded to make it of use to an unusually large range of readers, from advanced graduate students to seasoned researchers, and from topologists to the many other mathematicians who encounter singular spaces in geometrical, global analytical, or algebraic settings. It is therefore clearly intended to be readable at several different levels, with a superficial reading yielding an initial level of understanding, and with the necessary indications to geometrical topologists for a much closer reading. This is appropriate for a topic of large interest to a broad range of research communities. For a short book, to a surprising extent it achieves such varied goals. However, some workers in other related areas—e.g., operator algebras, algebraic K-theory, differential geometry, etc.—who beyond their general interest in the results and methods would like to attain full command of the topological details, will wish some of the discussions longer and more detailed.

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