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Complex dynamics and renormalization, by Curtis T. McMullen, Princeton University Press, 1994, vii + 214 pp., \$22.50, ISBN 0-691-02982-2 (cloth), 0-691-02981-4 (pbk)

Renormalization and 3-manifolds which fiber over the circle, by Curtis T. McMullen, Princeton University Press, 1996, vii + 253 pp., ISBN 0-691-01154-0 (cloth), 0-691-01153-2 (pbk)

To put McMullen's books into perspective, we should overview three big themes of modern holomorphic dynamics: rigidity, renormalization and fractal geometry.

A prototype for the first theme is the classical Mostow rigidity: two homotopically equivalent compact hyperbolic three dimensional manifolds are isometric. A classical proof consists of two steps. As first step, construct a quasi-isometry between the two manifolds and promote it to a quasi-conformal (qc) conjugacy between the corresponding Kleinian groups acting at the ideal boundary $\bar{\mathbb{C}} = \partial\mathbb{H}^3$. As second step, show that such a qc conjugacy must be conformal. Observe that otherwise the group would admit a measurable invariant tangent line field. Then blow up this line field near a Lebesgue point (where the line field is almost constant) by appropriate Möbius transformations and pass to a limit field invariant with respect to a limit of conjugate groups. But this limit field is too good (constant) to be invariant under a non-elementary group.

The dynamical counterpart of the Mostow rigidity would be a statement that two conformal dynamical systems which are combinatorially (or topologically, or quasi-conformally) equivalent are conformally equivalent. A link between the two settings is clear from the relation between hyperbolic manifolds and Kleinian groups. However, the realm of conformal dynamics is much bigger than the realm of Kleinian groups, and in general this link becomes more elusive.

A rich example is provided by the quadratic family $P_c : z \mapsto z^2 + c$ of dynamical systems in $\mathbb{C}$. The dynamical complexity of this family is encoded by the widely known Mandelbrot set M. The boundary of this set is the bifurcation diagram of the quadratic family. Outside ∂M (together with countably many special isolated parameter values) the maps are structurally stable, i.e., the dynamics changes continuously (even quasi-conformally) with c. Among these stable maps there is a special class of hyperbolic maps, i.e., the maps which have an attracting cycle. A central open problem in the field going back to Fatou is to show that actually any stable map is hyperbolic. This problem boils down to the qc rigidity problem: to show that any non-hyperbolic quadratic map P_c is qc rigid. In turn, this rigidity statement is equivalent to the absence of invariant measurable line fields on the Julia set $J(P_c)$. One can immediately see the relation with the second part of Mostow rigidity. The problem, though, is that a priori there is not enough expansion and compactness to carry out the blow-up argument.

There is an even stronger conjecture that any non-hyperbolic quadratic polynomial is combinatorially rigid. (Combinatorial classes are determined by the combinatorics of external rays landing at the periodic points of f. Keep in mind that a priori they are even bigger than the topological classes.) This rigidity conjecture

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turns out to be equivalent to the MLC Conjecture of Douady and Hubbard which would assert that the Mandelbrot set is locally connected. Importance of the MLC Conjecture lies in the fact that it would lead to the complete understanding of the topological structure of M.

The progress in the rigidity problem (and perhaps the setting of the problem itself) started in the early 80's with the Thurston rigidity theorem for postcritically finite maps; see [DH2], [MT] (a quadratic map is called postcritically finite if the orbit of the critical point is finite). From the modern viewpoint, it is a very simple result. But it marked the moment when the ideas of hyperbolic geometry were first introduced into dynamics.

The next significant step was made in the work of Yoccoz in the early 90's. That work established combinatorial rigidity of all quadratic maps which are "at most finitely renormalizable". The proof was based on a powerful technical tool called a *puzzle*. Among other useful things, the reader can find in McMullen's first book a brief introduction to the puzzle technique.

The assumption of the Yoccoz theorem naturally brings us to the realm of renormalization theory. Renormalization ideas came to dynamics from statistical physics in the mid 70's via the work of Feigenbaum [F] (see also Coullet and Tresser [TC]). The idea is to study the dynamical or parameter scaling laws by iterating a certain renormalization operator R acting in the space of dynamical systems. The space should be selected according to the situation. For instance, it can be the space of real unimodal maps with non-degenerate critical point or their complex analogues, quadratic-like maps. (A quadratic-like map is a double branched covering $f : U \to U'$ between two topological disks such that $U \Subset U'$.) The renormalization procedure consists of taking an iterate f^p of the map, restricting it to an appropriate domain, and rescaling, so that we obtain a map of the same class (to make the definition accurate, one has to impose some extra combinatorial assumptions, in particular, the assumption of connectivity of the Julia set $J(Rf)$).

Accordingly, quadratic-like maps can be classified as "renormalizable" and "non-renormalizable". It turns out that the renormalizable quadratics can be recognized by looking at the Mandelbrot set: a parameter value is renormalizable if it belongs to a canonical little copy of the Mandelbrot set [DH1]. Repeating this procedure, we come up with the dichotomy of "infinitely" and "at most finitely" renormalizable maps.

After the result of Yoccoz, the focus in the rigidity problem for quadratics naturally shifted to the infinitely renormalizable maps. It is the central theme of McMullen's first book. His main result proven there is the quasiconformal rigidity of infinitely renormalizable quadratics $f = P_c$ satisfying a certain geometric assumption (bounds on the geometry of the postcritical set). This assumption was known to be true for real parameter values, so that one concludes that any real infinitely renormalizable map is quasi-conformally rigid.

As we mentioned before, qc rigidity of the maps in question is equivalent to the absence of measurable invariant line fields on the Julia set. Following the analogy with Kleinian groups, McMullen proves it by blowing up the invariant line field near its Lebesgue point and passing to a limit line field invariant with respect to a limit of appropriately rescaled maps (passing to limits is possible due to the above mentioned geometric assumption). In the limit the line field becomes too regular (real analytic) to exist.

Later on, the combinatorial rigidity of real infinitely renormalizable quadratics (within the real quadratic family) was established in [L1]. The proof is based on further combinatorial and geometric analysis of the puzzle. It implies density of hyperbolic maps in the real quadratic family. (Note that qc rigidity implies a possibility to approximate the map by a complex but not necessarily real hyperbolic map.)

Let us now go back to renormalization. Feigenbaum introduced the renormalization operator in order to explain the "universal scaling law" in one parameter families of unimodal maps. He showed that the universality would follow from the existence of the hyperbolic fixed point of R with a one dimensional unstable manifold. Since then there has been a good effort to prove this conjecture which turned out to be very deep. The most exciting stage began in the mid 80's when the ideas of holomorphic dynamics, Teichmüller theory and hyperbolic geometry penetrated into the subject.

The first step was to complexify the renormalization operator. It was done in the work of Douady & Hubbard [DH1], where the operator was extended to the space of quadratic-like maps. In 1986 Sullivan [S1] formulated a program of catching the renormalization fixed point. The idea was to consider the quasi-conformal class $\mathcal{H}$ of the Feigenbaum quadratic map as a Teichmüller space supplied with a natural Teichmüller metric contracted by the renormalization. If one can only prove that the renormalization is *uniformly* contracting, the fixed point f_* would be in one's hands. Moreover, the orbits in $\mathcal{H}$ would exponentially converge to this fixed point, so that $\mathcal{H}$ would be a part of the stable manifold of f_*. Sullivan's program led to many deep insights and results including a priori bounds (i.e., precompactness of the orbit of f under the renormalization) and a Teichmüller theory of Riemann surface laminations (see [MS], [S2]).

In the early 90's McMullen suggested a different way to catch the fixed point (given a priori bounds). He introduced an important dynamical object called a *tower* which appears as a geometric limit of the rescalings of the original map. Formally speaking, it is just a two-sided orbit of the renormalization, $\{R^n f\}_{n=-\infty}^{\infty}$. It is easy to see that the quasiconformal rigidity of a tower yields (given a priori bounds) convergence of the renormalization orbits in $\mathcal{H}$ to a unique fixed point. A central result of McMullen's second book is such a rigidity theorem for towers. The reader would not be surprised that the proof follows the same lines as the proofs of the previously mentioned qc rigidity theorems: blowing up the invariant line field near its Lebesgue point and passing to a geometric limit using a priori bounds.

Along the way the third theme, fractal geometry, comes to the scene. The geometric result proven by McMullen is that the Julia set $J(f)$ of the Feigenbaum map is "hairy" at the critical point, i.e., blow-ups of $J(f)$ near the critical point converge (in the sense of the Hausdorff metric) to the whole complex plane. This is the key geometric property behind the tower rigidity.

Note. The term "hairiness" was suggested by Milnor [M] in the parameter context. Milnor conjectured that the Mandelbrot set is hairy near the Feigenbaum parameter value. This Milnor Conjecture was proven in [L2].

In the further chapters McMullen quantifies his geometric results to prove the exponential convergence of the orbits in $\mathcal{H}$ to the fixed point f_*. Namely he shows that the critical point 0 is a "deep point" of the Feigenbaum Julia set $J(f)$, which means by definition that $J(f)$ becomes hairy near 0 at the power rate. This property implies that the conjugacy between two Feigenbaum maps is $C^{1+\delta}$-conformal at 0, which readily yields the exponential convergence of the renormalizations to the fixed point. Altogether it is a delicate and beautiful geometric analysis.

A different approach to exponential convergence based on the Schwarz Lemma in Banach spaces (which does not use either Teichmüller theory or the deep points geometry) was carried in [L2]. This work also contains the first computer-free proof of the original Feigenbaum observation of universal parameter scaling. Roughly speaking, the works of Sullivan and McMullen took care of the dynamical part of the Renormalization Conjecture, while [L2] goes on to the parameter part.

Let us now go back to Mostow rigidity. Remember that it is concerned with compact manifolds. In general, a non-compact hyperbolic manifold can be deformed by deforming the conformal structure on its ideal boundary. However, as McMullen argues, it is "inflexible", i.e., the amount of the deformation decays exponentially as you go inside the convex core. This quantitative version of Mostow rigidity leads to a quantitative version of Thurston's hyperbolization theorem for 3-manifolds which fiber over the circle. In the renormalization context, it translates exactly into the results outlined above ($C^{1+\delta}$-conformality and exponential convergence).

This analogy makes deeper the Sullivan dictionary between Kleinian groups and iterates of rational maps which motivated McMullen's approach to the renormalization problem. For example, the renormalization operator corresponds in this dictionary to the re-marking of the manifold, and the renormalization fixed point corresponds to the hyperbolic structure on the manifold. (One can notice, though, that the explicit three dimensional constructions are not really essential for the proofs.)

Besides their research value, McMullen's books are a step in filling up a gap in the expositional literature on holomorphic dynamics. Unfortunately, there are too many unwritten or not adequately written works in this field. Contrary to that, McMullen's results are written carefully with the necessary background developed in closed form. There are many fine foundational points worked out in the books. Among them is the notion of the Carathéodory convergence of quadratic-like maps (though one can feel a bit uncomfortable to deal with the non-Hausdorff topology), compactness lemmas, and a notion of the renormalization which does not refer to the combinatorics of the external rays (with a surprising counterexample showing the difference between the two notions). All these make the books an excellent introduction to renormalization theory and a valuable source for reference.

You will have good time reading these books. It is fascinating to fly in the space over the Julia landscape, first overviewing the whole picture from the blue sky, then diving into the darkness of the convex core and watching the tiny Julia sets grow like a shadow of a tower at the sunset, eventually filling in the whole horizon.

Added in proof. J. Kahn has recently shown me his approach to the exponential convergence of the renormalizations (developed in 1994 but unpublished) based on a Teichmüller theory of branched Riemann surfaces.

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