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RECENT DEVELOPMENTS ON THE RICCI FLOW

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ABSTRACT. This article reports recent developments of the research on Hamilton's Ricci flow and its applications.

INTRODUCTION

One of the fundamental problems in differential geometry is to find canonical metrics on Riemannian manifolds. Here by canonical metrics we mean metrics of constant curvature in various forms. In turn, the existence of a canonical metric on a manifold often has important topological implications. A well-known example is the uniformization theorem for closed surfaces. On the other hand, to find a canonical metric on a given Riemannian manifold is often a very difficult problem. For finding metrics of constant scalar curvature, it is the well-known Yamabe problem (see R. Schoen's paper [Sc] and the references therein for details). For metrics of constant Ricci curvature, i.e., Einstein metrics, one needs to solve Einstein's equation, which is extremely difficult in general.

In this article we shall describe the Ricci flow, or the parabolic Einstein equation, introduced by Richard Hamilton in 1982 [Ha1] for producing Einstein metrics of positive scalar curvature and constant positive sectional curvature, and report on some of the more recent progress on the Ricci flow, especially the recent work of Hamilton which is focused on his program to understand Thurston's geometrization conjecture for three-manifolds using the Ricci flow.

In 1982, motivated by Eells and Sampson's work [ES] on the harmonic map flow in 1964, Hamilton introduced the Ricci flow

$$(1) \quad \frac{\partial}{\partial t} g_{ij} = -2R_{ij}.$$

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Notice that because of the minus sign in the front of the Ricci tensor in the equation, the solution metric to the Ricci flow shrinks in positive Ricci curvature direction while it expands in the negative Ricci curvature direction. For example, on the 2-sphere, any metric of positive Gaussian curvature will shrink to a point in finite time. Since the Ricci flow Eq.(1) does not preserve volume in general, one often considers the *normalized* Ricci flow defined by

$$(2) \quad \frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \frac{2}{n} r g_{ij},$$

where $r = \int R dV / \int dV$ is the average scalar curvature. Under this normalized flow, which is equivalent to the (unnormalized) Ricci flow (1) by reparametrizing in time t and scaling the metric in space by a function of t , the volume of the solution metric is constant in time. Note also that Einstein metrics (i.e., $R_{ij} = c g_{ij}$) are fixed points of Eq.(2).

In [Ha1], Hamilton showed that on a closed Riemannian 3-manifold M^3 with initial metric of positive Ricci curvature, the solution $g(t)$ to the normalized Ricci flow Eq.(2) exists for all time and the metrics $g(t)$ converge exponentially fast, as time t tends to the infinity, to a constant positive sectional curvature metric g_∞ on M^3 . In particular, such a M^3 is necessarily diffeomorphic to a quotient of the 3-sphere by a finite group of isometries. It follows that given any homotopy 3-sphere, if one can show that it admits a metric with positive Ricci curvature, then the Poincaré Conjecture would follow.

Note the above result of Hamilton in particular implies that on a closed 3-manifold with positive Ricci curvature, the curvature of the solution metric to the normalized Ricci flow is uniformly bounded in time t for $0 \leq t < \infty$. However, on a general closed 3-manifold, the solution of the normalized Ricci flow may develop singularities, meaning the curvature of the solution will become unbounded as time t approaches T , where $[0, T)$ is the maximal time interval for the existence of the normalized Ricci flow (2). For example, if we take a dumbbell metric on the 3-sphere S^3 with a neck like $S^2 \times B^1$, we expect the neck will shrink because the positive curvature in the S^2 direction will dominate the slightly negative curvature in the B^1 direction. As a result, we expect the neck will pinch off in finite time, which means a singularity developing in the Ricci flow (topologically, neck pinching would correspond to the prime decomposition of a 3-manifold).

Recall that Thurston's Geometrization Conjecture [T] states that every closed 3-manifold can be decomposed into pieces and each piece admits one of the 8 geometric structures, which would provide a link between the geometry and topology of 3-manifolds, analogous in spirit to the case of surfaces. In particular, the conjecture has as a special case the Poincaré Conjecture.

Hamilton's approach towards proving Thurston's Geometrization Conjecture is to study the Ricci flow on closed 3-manifolds and try to show that given any initial Riemannian metric, the Ricci flow evolves it to a geometric structure, after performing suitable geometrical surgeries. More precisely, the program is to divide the study of the Ricci flow on closed 3-manifolds into two parts. First, try to analyze singularities of the Ricci flow which develop in finite time well enough to enable one to perform geometric surgeries before the singularities occur, which will decompose the manifold, and then continue the solution (which will be nonsingular). Second, classify solutions to the normalized Ricci flow which exist for all time $t \in [0, \infty)$

and have uniformly bounded sectional curvature. (These are called *non-singular* solutions.)

The rest of the paper is arranged as follows. After reviewing some background materials in section 1, the short time existence result for the Ricci flow will be discussed in section 2. In section 3, we shall describe various convergence results of the normalized Ricci flow. In section 4, the recent result of Hamilton on classification of non-singular solutions to the normalized Ricci flow is stated. Section 5 describes the neck pinching phenomenon for the Ricci flow on a closed 4-manifold with non-negative isotropic curvature where the situation is simpler. In section 6, we state the Harnack estimate for the Ricci flow on a positively curved manifold, an important tool in analyzing singularities of the Ricci flow, and its important consequences. Finally, in section 7, we discuss singularities of the Ricci flow for closed 3-manifolds and explain how and why the study essentially reduces to the existence of a Harnack-type estimate on a general closed 3-manifold.

1. BACKGROUND MATERIAL

An important class of problems in Riemannian geometry is to understand the interaction between the curvature and topology on a differentiable manifold. A prime example of this interaction is the Gauss-Bonnet formula on a closed surface M^2 , which says

$$\int_M K dA = 2\pi \chi(M),$$

where dA is the area element of a metric g on M , K is the Gaussian curvature of g , and $\chi(M)$ is the Euler characteristic of M .

To study the geometry of a differentiable manifold we need an additional structure: the *Riemannian metric*. The metric is an inner product on each of the tangent spaces and tells us how to measure angles and distances infinitesimally. In local coordinates (x^1, x^2, \dots, x^n) , the metric g is given by $\sum_{i,j} g_{ij}(x) dx^i \otimes dx^j$, where $(g_{ij}(x))$ is a positive definite symmetric matrix at each point x . For a differentiable manifold one can differentiate functions. A Riemannian metric defines a natural way of differentiating vector fields: *covariant differentiation*. In Euclidean space, one can change the order of differentiation. On a Riemannian manifold the commutator of twice covariant differentiating vector fields is in general nonzero and is called the *Riemann curvature tensor*, which is a 4-tensor on the manifold.

For surfaces, the Riemann curvature tensor is equivalent to the *Gauss curvature* K , a scalar function. In dimensions 3 or more, the Riemann curvature tensor Rm is inherently a tensor. In local coordinates, it is denoted by R_{ijkl} , which is anti-symmetric in i and k and in j and l , and symmetric in the pairs $\{ij\}$ and $\{kl\}$. Thus, it can be considered as a bilinear form on 2-forms which is called the *curvature operator*. We now describe heuristically the various curvatures associated to the Riemann curvature tensor. Given a point $x \in M^n$ and 2-plane Π in the tangent space of M at x , we can define a surface S in M to be the union of all geodesics passing through x and tangent to Π . In a neighborhood of x , S is a smooth 2-dimensional submanifold of M . We define *the sectional curvature* $K(\Pi)$ of the 2-plane to be the Gauss curvature of S at x :

$$K(\Pi) = K_S(x).$$

Thus the sectional curvature K of a Riemannian manifold associates to each 2-plane in a tangent space a real number. Given a line L in a tangent space, we can average the sectional curvatures of all planes through L to obtain the *Ricci curvature* $Rc(L)$. Likewise, given a point $x \in M$, we can average the Ricci curvatures of all lines in the tangent space of x to obtain the *scalar curvature* $R(x)$. In local coordinates, the Ricci tensor is given by $R_{ik} = \sum_{jl} g^{jl} R_{ijkl}$ and the scalar curvature is given by $R = \sum_{ik} g^{ik} R_{ik}$, where (g^{ij}) is the inverse of the metric tensor (g_{ij}) .

Since the Ricci flow, which is the topic of this expository paper, lies in the realm of parabolic partial differential equations, where the prototype is the heat equation, we give below a brief review of the heat equation.

Let (M^n, g) be a Riemannian manifold. Given a C^2 function $u : M \rightarrow \mathbb{R}$, its Laplacian is defined in local coordinates $\{x^i\}$ to be

$$\Delta u = \text{tr}_g (\nabla^2 u) = g^{ij} \nabla_i \nabla_j u,$$

where $\nabla_i = \nabla_{\frac{\partial}{\partial x^i}}$ is its associated covariant derivative (Levi-Civita connection). We say that a C^2 function $u : M^n \times [0, T) \rightarrow \mathbb{R}$, where $T \in (0, \infty]$, is a solution to the heat equation if

$$\frac{\partial u}{\partial t} = \Delta u.$$

One of the most important properties satisfied by the heat equation is the maximum principle, which says that for any smooth solution to the heat equation, whatever pointwise bounds hold at $t = 0$ also hold for $t > 0$.

Theorem 1. (Maximum principle: pointwise bounds are preserved) *Let $u : M^n \times [0, T) \rightarrow \mathbb{R}$ be a C^2 solution to the heat equation on a complete Riemannian manifold. If $C_1 \leq u(x, 0) \leq C_2$ for all $x \in M$, for some constants $C_1, C_2 \in \mathbb{R}$, then $C_1 \leq u(x, t) \leq C_2$ for all $x \in M$ and $t \in [0, T)$.*

This property, which exhibits the smoothing behavior of the heat equation, follows from the following more general result.

Proposition 2. *Let $u : M^n \times [0, T) \rightarrow \mathbb{R}$ be a C^2 function satisfying*

$$\frac{\partial u}{\partial t} \leq \Delta u + X \cdot \nabla u,$$

where the laplacian and dot product are defined with respect to a time-dependent metric $g(t)$, and $X(t)$ is any time-dependent vector field. If $u(x, 0) \leq C$ for all $x \in M$, for some constant $C \in \mathbb{R}$, then $u(x, t) \leq C$ for all $x \in M$ and $t \in [0, T)$.

Idea of Proof. At a point (x, t) where u attains its maximum at time t , we have $\Delta u \leq 0$ and $\nabla u = 0$, which by the equation implies $\partial u / \partial t \leq 0$.

A lot of the results concerning the Ricci flow are proved using this version of the maximum principle. An exception is when one obtains pointwise bounds for the Ricci or sectional curvatures, where one uses a maximum principle for systems, where the solutions (e.g., Ricci tensor or curvature operator) are no longer functions but rather sections of a vector bundle. We refer the reader to [Ha2] for details.

2. THE SHORT TIME EXISTENCE FOR THE RICCI FLOW

Nonlinear heat equations first appeared in Riemannian geometry in 1964 when Eells and Sampson [ES] used the harmonic map heat flow to show that any map between closed Riemannian manifolds, where the range has negative sectional curvature, is homotopic to a harmonic map. It was the search for a heat flow for Riemannian metrics which led Hamilton [Ha1] to discover the Ricci flow in 1982. The basic idea of the Ricci flow is to try to improve a given Riemannian metric by evolving it by its Ricci curvature.

Given a differentiable manifold M , we say that a one-parameter family of metrics $g(t)$, where $t \in [0, T)$ for some $T > 0$, is a solution to the (unnormalized) Ricci flow if

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$$

at all $x \in M$ and $t \in [0, T)$. The minus sign in the equation makes the Ricci flow a *forward* heat equation as we shall see below. The factor 2 is simply for normalization purposes.

Note that in local geodesic coordinates $\{x^i\}$, we have

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{ipjq} x^p x^q + O(|x|^3).$$

Therefore

$$\Delta g_{ij}(0) = -\frac{1}{3} R_{ij}$$

where Δ is the standard euclidean laplacian. Hence the Ricci flow is like the heat equation for a Riemannian metric

$$\frac{\partial}{\partial t} g_{ij} = 6\Delta g_{ij}.$$

The practical study of the Ricci flow is made possible by the following short-time existence result.

Proposition 3. *Given any smooth compact Riemannian manifold (M, g_0) , there exists a unique smooth solution $g(t)$ to the Ricci flow defined on some time interval $t \in [0, \epsilon)$ such that $g(0) = g_0$.*

We remark that the Ricci flow is a weakly parabolic system where degeneracy comes from the gauge invariance of the equation under diffeomorphisms. Therefore, short time existence does not follow from general theory. Richard Hamilton's original proof of the short time existence was involved and used the Nash-Moser inverse function theorem. Soon after, D. DeTurck [De] substantially simplified the short-time existence proof by breaking the diffeomorphism invariance of the equation (which causes difficulty in directly applying standard theory to prove short-time existence).

On the other hand, when M is a complex manifold and the initial metric g_0 is Kähler, the Ricci flow is strictly parabolic. This is due to the fact that the gauge group of biholomorphisms is a much smaller group compared with the full diffeomorphism group. In fact, in the Kähler case the Ricci flow can be reduced to a strictly parabolic scalar equation of Monge-Ampère type (see [Ba] and [Ca1]). Hence the short-time existence follows easily.

3. CONVERGENCE RESULTS

Given that short-time existence holds for any smooth initial metric, one of the main problems concerning the Ricci flow is to determine under what conditions the solution to the normalized equation exists for all time and converges to a constant curvature metric. Results in this direction have been established under various curvature assumptions, most of them being some sort of positive curvature. Since the Ricci flow (1) does not preserve volume in general, one often considers, as we mentioned in the introduction, the normalized Ricci flow (2):

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \frac{2}{n} r g_{ij}.$$

Under this flow, the volume of the solution $g(t)$ is independent of time.

To study the long-time existence of the normalized Ricci flow, it is important to know what kind of curvature conditions are preserved under the equation. In general, the Ricci flow tends to preserve some kind of positivity of curvatures. For example, positive scalar curvature is preserved in all dimensions. This follows from applying the maximum principle, Proposition 2, to the evolution equation for scalar curvature R , which is

$$\frac{\partial}{\partial t} R = \Delta R + 2|R_{ij}|^2.$$

In dimension 3, positive Ricci curvature is preserved under the Ricci flow. This is a special feature of dimension 3 and is related to the fact that the Riemann curvature tensor may be recovered algebraically from the Ricci tensor and the metric in dimension 3. Positivity of sectional curvature is not preserved in general. However, the stronger condition of positive curvature operator is preserved under the Ricci flow. Recall that the Riemann curvature tensor may be considered as a self-adjoint map $Rm : \wedge^2 M \rightarrow \wedge^2 M$. We say that a metric g has positive (non-negative) curvature operator if the eigenvalues of Rm are positive (non-negative). We remark that positivity of curvature operator implies the positivity of the sectional curvature (and in dimension 3, the two conditions are equivalent). Finally in the Kähler case, the condition of positive holomorphic bisectional curvature, which is a weaker condition than positive sectional curvature, is also preserved (see [Ba] and [Mo]).

Although the condition of positive scalar curvature is preserved in all dimensions, no convergence results are known for metrics satisfying this condition except in dimension 2. In dimension 3 one expects necks to pinch off corresponding to the prime decomposition of a 3-manifold.

To illustrate what the Ricci flow can do, we first state Hamilton's 1988 result on surfaces [Ha3]. It is interesting to note that this result was obtained a few years after his celebrated work on 3-manifolds (see below), and the proof is, in some sense, even harder than the latter one.

Theorem 4. *Let M be a closed surface. Then for any initial metric g_0 on M , the solution to the normalized Ricci flow exists for all time. Moreover,*

1. *If the Euler characteristic of M is non-positive, then the solution metric $g(t)$ converges to a constant curvature metric as $t \rightarrow \infty$.*
2. *If the scalar curvature R of the initial metric g_0 is positive, then the solution metric $g(t)$ converges to a positive constant curvature metric as $t \rightarrow \infty$.*

For surfaces with non-positive Euler characteristic, the proof was based primarily on maximum principle estimates for the scalar curvature. The proof for the case of

metrics with positive scalar curvature is highly non-trivial and uses what are known as Entropy and Harnack estimates (both depend on having $R > 0$). Convergence in the remaining case of a compact surface with positive Euler characteristic and initial metric with scalar curvature changing sign was proved in [Ch] by extending the techniques in [Ha3]. Bartz, Struwe, and Ye [BSY] and Hamilton [Ha4] have both given new proofs of this result, the former based on the Aleksandrov reflection method and the latter based on an isoperimetric estimate.

We now turn our attention to Hamilton's famous work on 3-manifolds in 1982 [Ha1].

Theorem 5. (Positive Ricci topological spherical space form) *Let (M^3, g_0) be a closed Riemannian 3-manifold with positive Ricci curvature. Then there exists a unique solution to the normalized Ricci flow $g(t)$ with $g(0) = g_0$ for all time and the metrics $g(t)$ converge exponentially fast to a constant positive sectional curvature metric g_∞ on M^3 . In particular, M^3 is diffeomorphic to a spherical space form.*

As a consequence, such a 3-manifold M is necessarily diffeomorphic to a quotient of the 3-sphere by a finite group of isometries. It follows that given any homotopy 3-sphere, if one can show that it admits a metric with positive Ricci curvature, then the Poincaré Conjecture would follow. More generally, the Elliptization Conjecture would follow from showing that any closed 3-manifold with finite fundamental group admits a metric with positive Ricci curvature.

Prior to Hamilton's work, it was not known that a constant positive sectional curvature metric (or equivalently, Einstein metric in dimension 3) exists on a closed 3-manifold with positive Ricci curvature. (Of course, this would follow from the Poincaré conjecture and Spherical Space Form Conjecture.) In fact, it is very important and difficult to show the existence of an Einstein metric on a given manifold in general. Yau's solution of the Calabi conjecture gives a powerful method for finding Kähler-Einstein metrics of non-positive scalar curvature, while Hamilton's Ricci flow produces Einstein metrics of positive scalar curvature.

Remarks on the proof of the theorem: The proof involves obtaining certain strong a priori estimates for the curvature and its derivatives. To give the reader a taste of the type of estimates involved, we outline the main estimate for the Ricci curvatures below. Recall that a metric is Einstein if the Ricci tensor is proportional to the metric:

$$R_{ij} = \frac{1}{n} R g_{ij}.$$

When $n \geq 3$, this implies the scalar curvature R is constant. It is a special feature of dimension 3 that Einstein implies constant sectional curvature. In dimension 3, one of the main estimates is the following, which says that the metric is close to Einstein at points where the scalar curvature is large.

Lemma 6. *There exist constants $\varepsilon > 0$ and $C < \infty$ depending only on the initial metric such that*

$$\frac{|R_{ij} - \frac{1}{3} R g_{ij}|^2}{R^2} \leq C R^{-\varepsilon}.$$

We note that the left-hand side is scale-invariant (i.e., does not change when the metric is multiplied by a positive constant). The right-hand side is small when R is large. Thus, if we can prove $R_{\min} \rightarrow \infty$, then we would have a scale-invariant pointwise measure of the difference of the metric from one having constant sectional

curvature tend to zero uniformly: $|R_{ij} - \frac{1}{3}Rg_{ij}|^2 / R^2 \rightarrow 0$. To show that $R_{\min} \rightarrow \infty$ requires more estimates, and we refer the reader to [Ha1] for details. In addition, higher derivative estimates for the curvatures are required to prove convergence in C^∞ .

Recall that in all dimensions, the condition of positive curvature operator is preserved under the Ricci flow. In 1986 Hamilton [Ha2] proved the following

Theorem 7. *If (M, g_0) is a compact 4-manifold with positive curvature operator, then there exists a unique solution $g(t)$ with positive curvature operator to the normalized Ricci flow with $g(0) = g_0$ for all time $t \in [0, \infty)$ such that as $t \rightarrow \infty$ the metric $g(t)$ converges in C^∞ to a smooth metric g_∞ on M with constant positive sectional curvature.*

Again, the significance of this result is that the Ricci flow produced a constant positive sectional curvature metric on a compact 4-manifold with positive curvature operator. As a consequence, such a 4-manifold must be diffeomorphic to either S^4 or $\mathbb{R}P^4$. Hamilton has conjectured that the above long-time existence and convergence result holds in all dimensions. This is still unresolved for $n \geq 5$, although when the initial metric has sufficiently pointwise pinched positive sectional curvatures, convergence results have been obtained by G. Huisken [Hu], C. Margerin [Ma], and S. Nishikawa [Ni]. These results generalized the well-known differentiable sphere theorems; see [AM] for a history. In dimension 4, Hamilton's result has been generalized by H. Chen [C] to the case of 2-positive curvature operator, which means the sum of every two eigenvalues of the curvature operator is positive.

To a large extent, one should view the surface case as a special case of the Ricci flow on Kähler manifolds, since Riemann surfaces can be regarded as 1-dimensional complex manifolds. For convergence of the Ricci flow on compact Kähler manifolds, the first author [Ca1] proved the following result in 1985.

Theorem 8. *Let M be a compact Kähler manifold with definite first Chern class $c_1(M)$. If $c_1(M) = 0$, then for any initial Kähler metric g_0 , the solution to the normalized Ricci flow exists for all time and converges to a Ricci flat metric as $t \rightarrow \infty$. If $c_1(M) < 0$ and the initial metric g_0 is chosen to represent the negative of the first Chern class, then the solution to the normalized Ricci flow exists for all time and converges to an Einstein metric of negative scalar curvature as $t \rightarrow \infty$. If $c_1(M) > 0$ and the initial metric g_0 is chosen to represent the first Chern class, then the solution to the normalized Ricci flow exists for all time.*

The proof is partly based on the a priori estimates of Yau for Monge-Ampère equations. In case of $c_1(M) > 0$, the convergence is open even when M has positive holomorphic bisectional curvature. For more recent progress on this problem, see [Ca3] and [CH].

For the study of Ricci flow on noncompact Kähler manifolds, W.-X. Shi has done very nice work. In [Sh2], Shi used the Ricci flow to prove that a complete non-compact Kähler manifold X^n with bounded and positive holomorphic bisectional curvature must be biholomorphic to \mathbb{C}^n , provided the manifold has euclidean volume growth and the average scalar curvature has quadratic decay. This provides a partial affirmative answer to a conjecture of Yau. In [Sh3], he dropped the condition of Euclidean volume growth and assumed that the average scalar curvature decays like $1/R^{1+\epsilon}$ and concluded that the manifold X^n is biholomorphic to a pseudoconvex domain in \mathbb{C}^n .

4. NON-SINGULAR SOLUTIONS ON 3-MANIFOLDS

As we have seen in the last section, the Ricci flow had important topological consequences for closed 3-manifolds of positive Ricci curvature. Now we turn our attention to the study of the Ricci flow on general closed 3-manifolds.

Although long-time existence and convergence hold for the Ricci flow on compact 2-dimensional manifolds with arbitrary initial metrics, this is certainly not the case in dimensions 3 and higher, where singularities develop in finite time for certain initial metrics. Here, by singularities of the Ricci flow we mean solutions to the normalized Ricci flow (2) with unbounded curvature. A typical example is when the manifold, say S^n , is shaped like a dumbbell. That is, there are two larger spherical regions to the left and right joined by a cylindrical region (the ‘neck’) in the middle. The topology of the neck is that of a cylinder $S^{n-1} \times [-1, 1]$. When $n \geq 3$, the sphere S^{n-1} has positive intrinsic curvature, which makes the neck shrink (provided the neck doesn’t open up too fast), leading to a singularity. Therefore the study of the Ricci flow on general 3-manifolds is much more complicated and difficult.

However, one can divide the study of the Ricci flow on 3-manifolds into two parts. First, try to analyze the singularities which develop in finite time well enough to enable one to perform geometric surgeries before the singularities occur, which will decompose the manifold, and then continue the solution. Second, classify solutions to the normalized Ricci flow which exist for all time $t \in [0, \infty)$ and have uniformly bounded sectional curvature (these are called *non-singular* solutions). In dimension 3 much progress has been made by Hamilton in both directions, especially the second part of the program, where Hamilton [Ha5] recently proved that non-singular solutions have geometric decompositions, which we shall describe below.

A solution (M, g) to the Ricci flow (normalized or unnormalized) on a time interval $[0, T)$ is said to be *maximal* if it cannot be extended past time T . In this case either

1. $T = \infty$ and $\sup_{M \times [0, T)} |Rm| < \infty$, or
2. $\sup_{M \times [0, T)} |Rm| = \infty$.

In the first case we say that the solution is *non-singular*, and in the second case we say that the solution is *singular* and that a singularity occurs at time $T \leq \infty$.

4.1. Examples. By the results stated in the previous sections, the normalized Ricci flow on a closed manifold is non-singular when the initial metric is

- N1. any metric on a surface
- N2. a metric on a 3-manifold with positive Ricci curvature
- N3. a locally homogeneous metric on a 3-manifold (see Isenberg-Jackson [IJ])
- N4. a metric on a 4-manifold with positive curvature operator
- N5. a metric on an n -manifold with sufficiently pointwise-pinched positive sectional curvatures
- N6. a Kähler metric on a compact complex n -manifold with first Chern class $c_1 = 0$ or $c_1 < 0$

In each case, we actually have convergence to an Einstein metric as $t \rightarrow T$. In contrast, the unnormalized Ricci flow on a compact manifold is singular when the initial metric is

- U1. any metric on a surface with $\chi > 0$
- U2. a metric on an n -manifold with positive scalar curvature

- U3. a locally homogeneous metric on a 3-manifold of class $SU(2)$ or $S^2 \times \mathbb{R}$
- U4. a Kähler metric on a compact complex n -manifold with $c_1 > 0$

Note that in cases U1 and U3 the singularity is *removable* by normalizing the flow. The same is true in case U2 if the initial metric is as in either case N2, N4 or N5.

4.2. Non-singular solutions. In dimension 3, non-singular solutions are now well-understood topologically through Hamilton's work [Ha5]. In particular, closed 3-manifolds which admit a non-singular solution can also be decomposed into geometric pieces. This focuses the study of the Ricci flow on closed 3-manifolds on the analysis of singularities. Now we present the main result in Hamilton's work [Ha5].

Theorem 9. *If a closed differentiable 3-manifold M^3 admits a non-singular solution $g(t)$ to the normalized Ricci flow, i.e., a solution which exists for all time and has uniformly bounded sectional curvature, then M^3 has a decomposition into geometric pieces. In particular, M^3 is diffeomorphic to either*

1. a Seifert fibered space
2. a spherical space form S^3/Γ
- 3a. a flat manifold
- 3b. a hyperbolic (constant negative sectional curvature) manifold
4. the union along incompressible tori of finite volume hyperbolic manifolds and Seifert fibered spaces

Hamilton actually proves the following. If $(M^3, g(t))$ is a non-singular solution to the normalized Ricci flow on a closed 3-manifold, then exactly one of the following occurs:

1. (Sequential collapse) There exists a sequence of times $t_i \rightarrow \infty$ such that the metrics $g(t_i)$ collapse, i.e.,

$$\lim_{i \rightarrow \infty} \rho(t_i)^2 \max_{M \times \{t_i\}} |Rm| = 0,$$

where $\rho(t_i) = \max_{x \in M} \text{inj}(x, t_i)$ is the maximum injectivity radius of any point for the metric $g(t_i)$. By Cheeger-Gromov theory [CG], M^3 admits an F -structure and is topologically a graph manifold. This implies M^3 is a Seifert fibered space. (See [A1] for an exposition of Cheeger-Gromov theory.)

2. (Exponential convergence to a spherical space form) The solution $g(t)$ converges exponentially fast in every C^k -norm to a constant positive sectional curvature metric g_∞ on M^3 . This implies M^3 is diffeomorphic to a space form S^3/Γ .
3. (Sequential convergence to a flat or hyperbolic manifold) There exists a sequence of times $t_i \rightarrow \infty$ and self-diffeomorphisms ϕ_i of M^3 such that the sequence of pulled back metrics $\phi_i^* g(t_i)$ converge to a constant (either zero or negative) sectional curvature metric g_∞ on M^3 as $i \rightarrow \infty$.
4. (Toroidal decomposition into hyperbolic and Seifert fibered pieces) There exist a finite collection of complete noncompact 3-manifolds $\{H_\alpha^3, h_\alpha\}$ with constant negative sectional curvature and finite volume and smooth 1-parameter families of diffeomorphisms (into) $\psi_\alpha(t) : H_\alpha^3 \rightarrow M^3$ such that the pulled-back metrics $\psi_\alpha(t)^* g(t)$ converge to h_α as $t \rightarrow \infty$. Moreover, M^3 can be decomposed into two time-dependent 3-manifolds $M_1(t)$ and $M_2(t)$ ($M_j(t)$ and $M_j(t')$ are isotopic for all t and t' , $j = 1, 2$) whose intersection is their

mutual boundary, which consists of a finite disjoint union of 2-tori. Geometrically, the metrics pulled-back from $M_1(t)$ converge to the hyperbolic pieces, i.e., $(\psi_\alpha^{-1}M_1, \psi_\alpha^*g)(t)$ converge to $\cup_\alpha (H_\alpha^3, h_\alpha)$ as $t \rightarrow \infty$, while $(M_2(t), g(t))$ collapses. Topologically, the boundary tori are all incompressible, and all of the homomorphisms induced by the inclusion maps $i_* : \pi_1(N) \rightarrow \pi_1(P)$, where $N = M_1(t) \cap M_2(t)$, $M_1(t)$ or $M_2(t)$ and $P = M_1(t)$, $M_2(t)$ or M , are injective.

The most difficult case is the last one where hyperbolic limits occur. To show that $\pi_1(H_i)$ injects into $\pi_1(M)$ for each hyperbolic limit, Hamilton uses a minimal surface argument, part of which is reminiscent of Schoen and Yau's [SY] use of minimal surfaces to classify compact 3-manifolds with positive scalar curvature up to homotopy type (and modulo the spherical space-form conjecture). The idea is to represent a non-trivial element of the kernel by a minimal disk and show that the rate of change of the area of the minimal disk under the Ricci flow is uniformly bounded from above by a negative constant, leading to a contradiction since the initial disk has finite area. The main estimate is to show that the length of the boundary of the minimal disk bounds the area of the disk. Hamilton also proves that M may be written as the union of two manifolds (depending on time) $M_1(t)$ and $M_2(t)$ whose intersection is their mutual boundary which is a finite disjoint union of tori such that $(\psi_i(t)^{-1}[M_1(t)], \psi_i(t)^*g(t))$ converges to $\cup_{i=1}^k (H_i, h_i)$ as $t \rightarrow \infty$, and $(M_2(t), g(t))$ collapses as $t \rightarrow \infty$.

5. NECKS IN A 4-DIMENSIONAL CASE

The first part of Hamilton's program, the analysis of singularities, is more difficult than the second. However, in dimension 4 some aspects of it are simpler than in dimension 3. Recently Hamilton [Ha6] has classified compact 4-manifolds with non-negative *isotropic curvature* using the Ricci flow where singularities (pinching necks) develop in finite time. A manifold of dimension $n \geq 4$ has non-negative isotropic curvature if and only if for every orthonormal 4-frame $\{e_i\}_{i=1}^4$, we have

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} \geq 2R_{1234},$$

where $R_{ijkl} = \langle R(e_i, e_j)e_l, e_k \rangle$. In 1988 Micallef and Moore [MM] showed that a simply-connected compact n -manifold with positive isotropic curvature is homeomorphic to S^n by studying the index of minimal 2-spheres. In dimension 4, Hamilton relaxed the condition that M be simply-connected and also obtained a classification up to diffeomorphism. In particular, he proved

Theorem 10. *Let M be a compact 4-manifold with no essential incompressible space-form, that is, with the property that if N is a submanifold of M diffeomorphic to a space-form S^3/Γ such that $i_* : \pi_1(N) \rightarrow \pi_1(M)$ is an injection, then Γ is isomorphic to either $\{1\}$ or \mathbb{Z}_2 . If M admits a metric with non-negative isotropic curvature, then M is diffeomorphic to either S^4 or the connected sum of copies of $\mathbb{R}P^4$, $S^3 \times S^1$, and $S^3 \widetilde{\times} S^1$ (the unique non-orientable S^3 bundle over S^1 .) The converse is also true.*

The proof of this result uses a difficult analysis of the system of PDEs satisfied by the Riemann curvature operator to show that the only singularities which can develop are necks pinching off. Here a neck is a (usually small) piece of the manifold which metrically is close to a quotient by isometries of a constant multiple of the standard metric on a long thin finite cylinder $S^3 \times [-L, L]$, where $L \gg 1$ (one

also needs to consider \mathbb{Z}_2 quotients). At a suitable time before a singularity forms, Hamilton geometrically performs a surgery at the neck which either is of the form $S^3 \times B^1 \rightarrow B^4 \times S^0$ or removes a $\mathbb{R}P^4$ summand and then discards any component of the new manifold which is diffeomorphic to either S^4 , $\mathbb{R}P^4$, $S^3 \times S^1$, or $S^3 \widetilde{\times} S^1$. He then continues the Ricci flow starting with the new 4-manifold and metric and shows that after only a finite number of surgeries one obtains the empty set. One of the basic ingredients in analyzing the singularities and performing the surgeries is the Harnack estimate.

6. THE HARNACK ESTIMATE

Harnack inequalities are fundamental in the study of elliptic and parabolic partial differential equations. In the Ricci flow it is basic in the understanding of singularities. In [Ha7] Hamilton proved a complicated *differential* Harnack inequality for the Ricci flow on n -manifolds using the method of Li and Yau developed in [LY] in 1986.

Theorem 11. (The Harnack estimate for the Ricci flow) *If $(M, g(t))$ is a solution to the Ricci flow on a compact manifold with non-negative operator, then for any vector field W and 2-form U , we have*

$$Z = M_{ij}W^iW^j + 2P_{ijk}U^{ij}W^k + R_{ijkl}U^{ij}U^{kl} \geq 0,$$

where

$$P_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik}$$

and

$$M_{ij} = \Delta R_{ij} - \frac{1}{2}\nabla_i \nabla_j R + 2g^{kp}g^{lq}R_{ikjl}R_{pq} - g^{kl}R_{ik}R_{jl} + \frac{1}{2t}R_{ij}.$$

In the Kähler case, a similar Harnack estimate was proved by the first author [Ca2] under the weaker curvature assumption of non-negative holomorphic bisectional curvature. See [CC] for a geometric interpretation of Hamilton's Harnack estimate. As we shall see below, the Harnack estimate is important because it is the major tool in analyzing singularities of the Ricci flow.

The first important consequence of the Harnack estimate is to relate singularities of the Ricci flow to Ricci solitons, which are special solutions of the Ricci flow moving along the equation by diffeomorphisms, that is, where the metric $g(t)$ is the pull-back of the initial metric $g(0)$ by a 1-parameter family of diffeomorphism $\phi(t)$ generated by a vector field on manifold M . (In the PDE literature, they are more commonly referred to as self-similar solutions.) It turns out that on a Ricci soliton, the Harnack-Li-Yau quadratic Z in the above theorem vanishes. Using the Harnack estimate, one can show that under certain conditions, limits of dilations (i.e., blow up) of singularities of the Ricci flow are (gradient) Ricci solitons (see [Ha8] and [Ca3]). Hence, the study of Ricci solitons also becomes important.

The equations that characterize a Ricci soliton metric are given by $\nabla_i W_j = \nabla_j W_i = R_{ij}$, where $W = \{W^i\}$ is a vector field on M . If W is a gradient vector field, then we have a *gradient* Ricci soliton. The first example of a *steady* gradient Ricci soliton was found by Hamilton [Ha3] in dimension 2. It is defined on \mathbb{R}^2 and has the form

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}.$$

This Ricci soliton is called the *cigar soliton* Σ , because it is asymptotic to a flat cylinder at infinity and has maximal curvature at the origin. Higher dimensional examples of rotational symmetric Kähler-Ricci solitons have been found recently by the first author (see [Ca3] and [Ca4]).

Another important consequence of the Harnack estimate is the following *Little Loop Lemma* for 3-manifolds with non-negative sectional curvature (see [Ha9], section 15). Roughly speaking, it states that there is no little geodesic loop in a large flat region. More precisely, we have

Lemma 12 (Little Loop Lemma). *Let $(M, g(t))$ be a solution to the Ricci flow on a compact 3-manifold with non-negative sectional curvature. There exists a universal constant $A > 0$ and a constant $B > 0$ depending only on the initial metric g_0 such that if x is a point where $R(y, t) \leq \frac{A}{\rho^2}$ for all $y \in B_\rho(x)$ at some time t and for some ρ , then we have the injectivity radius estimate*

$$\text{inj}(x) \geq B\rho,$$

at that time t .

As we shall see in the next section, the Little Loop Lemma can be used to rule out certain types of singularities.

Hamilton conjectures that this result holds in all dimensions under the assumption of non-negative curvature operator.¹

7. SINGULARITIES IN DIMENSION 3

One reason that the Harnack estimate is relevant, even under the restrictive assumption of non-negative curvature operator, is that in dimension 3, limits arising from dilating singularities have non-negative sectional curvature (which is the same as non-negative curvature operator when $n = 3$). In particular the Harnack estimate is one of the estimates used in proving the following classification of 3-dimensional singularities given by Hamilton in [Ha9].

Proposition 13. *If $(M, g(t))$ is a solution to the Ricci flow on a compact 3-manifold where a singularity develops in finite time T , then there exists a sequence of dilations of the solution which converges to a quotient by isometries of either S^3 , $S^2 \times \mathbb{R}$, or $\Sigma \times \mathbb{R}$, where Σ is the cigar soliton.*

Hamilton also conjectured that the limits which are quotients of $\Sigma \times \mathbb{R}$ cannot occur and outlined an approach for proving this. If this is true, then there always exists a sequence of dilations which converge to a quotient either S^3 or $S^2 \times \mathbb{R}$. In the first case, the manifold has to be a topological space form. In the second case, one sees a neck, on which one wants to perform a surgery. One would then like to show that after a finite number of surgeries, the solution becomes a non-singular solution.

To see how one might be able to rule out limits which are quotients of $\Sigma \times \mathbb{R}$, notice that at its infinity, Σ is asymptotic to a cylinder; hence the limits which are quotients of $\Sigma \times \mathbb{R}$ have short (geodesic) loops in relatively flat regions and so does the manifold right before taking the limits. Now for a solution to the Ricci flow on a compact 3-manifold which has non-negative sectional curvature, these short geodesic loops can be ruled out by the Little Loop Lemma stated in the last section.

¹The proof in [Ha9] for all dimensions is apparently not complete. However, Hamilton has announced a complete proof when $n = 3$.

The problem is that even though the limits of dilations have non-negative sectional curvature, the solution metric right before taking the limit does not have non-negative sectional curvature, hence the Harnack estimate and Little Loop Lemma do not apply directly. Thus we need to have the Little Loop Lemma for solutions to the Ricci flow with possible small negative curvature somewhere, as in the case right before taking the limit of dilations.

In conclusion, the limits $\Sigma \times \mathbb{R}$ and $\Sigma \times S^1$ can be ruled out in general if the following conjecture of Hamilton is true.

Conjecture 14. A Harnack-type estimate holds for the Ricci flow on compact 3-manifolds with arbitrary initial metric.

Thus proving this conjecture is a crucial step in Hamilton's program to understand Thurston's geometrization conjecture. One reason for believing that this conjecture may be true is the following estimate saying that in a sense the sectional curvatures tend to positive (see [Ha9] or [Iv]).

Proposition 15. *Let $(M, g(t))$ be a solution to the Ricci flow on a compact 3-manifold. There exist constants $C < \infty$ and $c > 0$ depending only on the initial metric g_0 such that if at some point (x, t) the scalar curvature $R(x, t) \geq C$, then the minimum sectional curvature K_{\min} satisfies*

$$K_{\min} \geq -c \frac{R}{\ln R}$$

at (x, t) .

Thus, if the scalar curvature R is large at a point, then $|K_{\min}|$ is much smaller than R at that point.

There are also many other important works on the Ricci flow which we have not discussed in this article, including the works by Bemelmans-Min-Oo-Ruh, R. Bryant, Cafora-Isenberg-Jackson, T. Ivey, D. Knopf, N. Koiso, Leviton-Rubinstein, Min-Oo, Y. Shen, L.-F. Wu, D. Yang, and R. Ye, to name a few. Finally we mention that another approach to the Geometrization Conjecture has been studied by M. Anderson [A2].

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