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Moduli of curves, by J. Harris and I. Morrison, Springer, New York, 1998, 366 pp., \$59.00, ISBN 0-387-98438-0

Every mathematician knows about Riemann surfaces, i.e. complex varieties of dimension one, and their ubiquitous presence in many branches of our science. The theory of Riemann surfaces belongs, strictly speaking, to complex analysis. However, as discovered by Riemann, compact Riemann surfaces can be actually embedded in some projective space as subvarieties defined by a set of homogeneous polynomial equations. In other words, they can be concretely realized as algebraic curves. Taking this point of view, the theory of compact Riemann surfaces coincides with the one of curves in projective spaces, and therefore it can be viewed as a part of algebraic geometry.

Though the study of properties of single compact Riemann surfaces is an important and charming subject, probably the most fascinating aspect of the theory is the fact that compact Riemann surfaces naturally vary in families, and therefore properties of specific members can be better understood when compared with analogous properties of general members of the families in question. More precisely, compact Riemann surfaces are, from a topological viewpoint, compact oriented surfaces, and as such they possess a basic invariant, the *genus*. Riemann surfaces with the same genus are homeomorphic, actually diffeomorphic. However, they are not, in general, isomorphic as complex varieties. Actually, as discovered by Riemann, the complex structure of Riemann surfaces of genus $g \geq 2$ varies with $m_g = 3g - 3$ complex parameters or *moduli*, whereas the number of moduli m_g is g if $g = 0, 1$.

What Riemann meant with this assertion is that there is a complex variety $\mathcal{M}_g$ of dimension m_g whose points are in one-to-one correspondence, in some natural way, with the isomorphism classes of compact Riemann surfaces of genus g. However, neither Riemann nor any of his followers for about one century—among these the algebraic geometers of the classical Italian school like Castelnuovo, Enriques, Severi, etc.—ever proved this assertion. Rather, with a visionary attitude and a remarkable intellectual courage, they started pioneering the new area. Usually with inadequate techniques and therefore with unsatisfactory and incomplete arguments, they tried to prove theorems and went on making conjectures on $\mathcal{M}_g$—as well as on related objects like the so-called Severi variety $V_{d,g}$, i.e. the family of irreducible plane nodal curves of degree d and genus g—as if they really knew how to construct these objects and to deduce their main properties. Among the most famous statements they made, it is worth mentioning by way of example the following two, both due to Severi: (i) *Severi's conjecture*, which asserts that $\mathcal{M}_g$ is unirational for all g; (ii) the so-called *Severi's theorem*, which says that $V_{d,g}$ is irreducible for all d and g. The former statement, by the way, was important in Severi's mind, since it implies an algebro-geometric proof of the irreduciblity of $\mathcal{M}_g$. A lot of the motivation for the material presented in this book resides in the successfull attempt made in the last decades in order to shed light, among other things, on both Severi's conjecture, which has been disproven, and Severi's theorem, which by contrast has been proved

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to hold, though Severi's original argument was substantially incomplete. In fact, not by chance, the very last pages of the book are devoted just to these topics.

There are several beautiful, classical and more recent books which treat the basic theory of compact Riemann surfaces and algebraic curves—too many to be mentioned here. On the other hand, both classically and recently, there have been very few authors who accepted the challenge of treating more advanced topics concerning algebraic curves, like moduli spaces. The most famous classical book is certainly the treatise of F. Enriques and O. Chisini, *Teoria geometrica delle equazioni e delle funzioni algebriche*, in four volumes, edited in the course of about twenty years, from 1915 to 1934. In this book there is a complete collection of the classical knowledge and techniques on the subject, which starts from the basics and goes to the forefront of research at the time. Nothing comparable to the Enriques-Chisini treatise appeared in recent times, till the publication in 1985 of the *four authors' book* by E. Arbarello, M. Cornalba, Ph. Griffiths and J. Harris, *Geometry of algebraic curves*, vol. I, which treats the advanced theory of algebraic curves and their maps to projective spaces, i.e. of linear series on curves. That book does not deal, however, with the moduli space, a subject to which a second volume, already announced, will be dedicated. This topic is instead the object of the present book, which we believe, together with the four authors' book, will become the main tool for all those who want to enter into the subject and really learn how to do research in this beautiful, classical area of mathematics.

It is perhaps worthwhile, however, to stress right away the main differences between the two books. This will give us in fact the opportunity to briefly illustrate the viewpoint of the authors of the present one, though we will of course return to this matter later when we discuss the contents of the book. The four authors' book, which has to be naturally considered as the main prerequisite for reading the present one, is intended to be a "comprehensive and self-contained account"—we quote the preface of that book here—of the theory of linear series on curves. Moreover it is an excellent, systematic reference book on the topics which have been treated there. By contrast, the present book—again we quote from the preface—"isn't intended to be a definitive reference"; the preference of the authors "has been to focus on examples and applications rather than on foundations ... to sacrifice proofs of some, even basic, results ... in order to show how the methods are used to study moduli of curves." Having all this in mind, we can now go over the contents of the book.

Chapters 1 and 2 provide a non-systematic guidebook to some basic material concerning Hilbert schemes and related matters, like Severi varieties and Hurwitz schemes (chapter 1), and moduli spaces (chapter 2). More precisely, in chapter 1 the reader will find the essential ingredients involving families of projective curves which will be used in the rest of the book. Chapter 2 has in principle the same purpose in relation to the moduli space of curves. However, in practice, it works in a dual way: on one side it introduces the basic concepts and topics which occupy the rest of the book; on the other it also quickly mentions and explains important ideas and results which will not be used later on, but which the authors do not feel like leaving totally aside. Among these: (i) various constructions of moduli spaces, like the Teichm*ü*ller's approach or the Hodge theoretic approach, while the geometric invariant theoretic approach will be the only one really used later; (ii) cohomology of $\mathcal{M}_g$ and J. Harer's theorems, based on Teichm*ü*ller's approach, and all related material like Mumford's standard conjectures on the stable cohomology ring of

$\mathcal{M}_g$, cohomology of Hilbert schemes and Enriques-Franchetta's conjecture, Faber's conjectures on the structure of the Chow ring of $\mathcal{M}_g$; (iii) recent developments, related to physics and quantum cohomology, like Witten-Kontsevich's formula and moduli spaces of stable maps, with enumerative applications to calculations of degrees of Severi varieties, etc. Chapters 1 and 2 cover, on the whole, the first eighty pages of the book. They should be pleasant reading for all who want to briefly become acquainted with the main ideas in the field and what has been done or is going on. Beginners instead will have a hard time if they really want to understand every detail and if they want to work out all the exercises in which often essential parts of proofs and calculations have been confined. For them an advisory is in order: for the material covered by chapter 1 they can get excellent help from E. Sernesi's book *Topics on families of projective curves*, Queen's Papers in Pure and Applied Math., **73** (1986), which unfortunately has not been quoted here.

Chapter 3 develops, in a more systematic way, the main technical tools which are needed to work with families of curves and moduli: dualizing sheaves, deformations and specializations, Grothendieck-Riemann-Roch theorem and related enumerative computations, admissible covers, etc. In particular the part on stable reduction, which is an essential concept for whoever wants to work in the field, is based on the discussion of a few critical examples and turns out to be very illuminating. An interlude on the moduli stack is included. There, without entering into too many technicalities, the main motivations for the use of stacks in the context of moduli spaces of curves and the main ideas in the theory of stacks are introduced, discussed and used. Those who want to work on this subject will certainly find here stimulus for further reading.

Chapter 4 contains the construction of $\overline{\mathcal{M}}_g$ based on ideas of Mumford and Gieseker using geometric invariant theory. The first part of the chapter is devoted to a quick introduction to the main ideas and results of geometric invariant theory, the second part to stability of curves, the third to the construction of $\overline{\mathcal{M}}_g$ and to the proof that it is a projective scheme. It is nice to have all this material collected here in fifty pages. However, again, working out all the details, some of them presented as exercises, will not be easy for the generic reader.

The book culminates with chapters 5 and 6, which contain most of the developments of the theory worked out or inspired by the first author of the book. They contain the material whose exposition has apparently been the real purpose of the book itself, and, not by chance, they turn out to be particularly interesting and enjoyable.

In chapter 5 the theory of limit linear series on tree-like curves is presented. This is the theory of how linear series on a general curve degenerate when the curve itself degenerates to a reducible one, which has to be, in order to better control such specializations, simple enough. This theory was founded by D. Eisenbud and J. Harris in the early '80's and has proved to be very effective in many problems concerning properties of general curves. For instance, in this framework, the famous theorem of Brill-Noether about the dimension of families of linear series on a general curve becomes a surprisingly easy application, whereas the also famous theorem of Gieseker-Petri, which gives information about the local structure of the aforementioned families, is a less easy, but still reasonably manageable application of this theory. Both results can be found in this chapter.

Finally, in chapter 6 the authors collect a number of important results whose not easy proofs are presented in a reasonably complete, though friendly, way, often exploiting interesting shortcuts with respect to the original papers. The results in question include: (i) Deligne-Mumford's theorem on the irreducibility of $\mathcal{M}_g$; (ii) Diaz's theorem on complete subvarieties of $\mathcal{M}_g$; (iii) divisors on $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ and their ampleness properties; (iv) the irreducibility of the Severi varieties; (v) the proof that $\mathcal{M}_g$ is of general type for $g \geq 24$ and has Kodaira dimension at least 1 for $g = 23$, a result which disproves Severi's conjecture mentioned before. Often open questions and suggestions for further research are presented and discussed here.

Let me conclude now with some remarks and personal opinions. The book, as the authors say in the preface, "is based on notes from a course at Harvard in 1990." Therefore it often reflects the informal style of lectures in which the reader is thrown, as soon as possible, into deep waters. The authors actually help him to learn how to swim if he is really able to do so, but also let him face right away all the difficulties of the subject. Thus, the style turns out to be pleasant without concealing the difficulties, and it is clear they make an effort to make introductions of techniques as natural as possible and to illustrate connections between various aspects of the theory. However, it also has some disadvantages. Sometimes the reader could be unsure about what has been proved at a given point and what is still left to be proved. Moreover, often essential parts of some proofs are left as exercises, which are far from being—as an exercise should in principle be—a more or less immediate application of the theory treated so far. This attitude starts at the beginning of the book: to mention one of the first examples, look at exercise (1.46), pg. 28, in which some of the basics of Brill-Noether theory are laid down. Then it continues, as I have already remarked in describing the contents of the book, with some peaks like, for instance, the proof of the potential stability theorem in chapter 4, pgs. 224-234, in which a few non-trivial steps are left as exercises.

Speaking of exercises, I am tempted to express here some more objections about the way they are used. A few of them, for instance, are, to the best of my knowledge, still open research problems, and some are even rather hard ones! Just to mention one example, think about exercise (2.22) on pg. 51. The first question asks the reader to compute the number of so-called "poligonal" curves of genus $g \leq 5$; this is not difficult. The final question, in contrast, asking the reader to compute or guess—this is not really specified in the text—the above number for any g is a hard, open combinatorial problem, for which only an asymptotic answer is known in general (see A. A. Bender, R. Canfield, *The asymptotic number of labeled graphs with given degree sequence*, J. of Combinatorial Theory, A **24** (1978), 296-307). A reader who is unable to solve this exercise as well as others (but not all; it is his problem to understand which!) should not think of himself as foolish. Exercise (1.43), concerning the Hilbert scheme of Castelnuovo's curves, presents a different kind of pitfall. The problem presented there is in fact part of a more general question examined by the reviewer in the paper *On the Hilbert scheme of curves of maximal genus in a projective space*, Math. Z. **194** (1987), 351-363, which is not quoted here. Nothing bad in this: the authors warned us that this is not a reference book. Moreover, this is a case of a simple exercise, which the reader should really be able to solve on his own. However, the danger is that a reader who is a beginner, and therefore not aware of the current literature, could be deceived about what is still an open research problem and what is not. Finally, one more case, at the very end of the book—only the more strenuous and motivated readers should have arrived there! Here they find exercise (6.78) on pg. 342, in which they are asked to compute the class in $\overline{\mathcal{M}}_g$ of the locus of curves C with a semicanonical pencil, i.e.—we quote from the text—"a line bundle L with $L^{\otimes 2} \simeq \omega_C$ and $h^0(C, L) \geq 2$." The point is that this locus, as defined in the book, is not purely dimensional: it has, in general, a divisorial part, which is the closure of the locus of curves C with a line bundle L such that $L^{\otimes 2} \simeq \omega_C$ and $h^0(C, L) = 2$, and a part of codimension 3. Since the title of the paragraph recites "Further divisor class calculation", probably the authors meant the reader should compute the class of the divisorial part of the locus in question. Or do they want to test the degree of alertness of the surviving reader?

In any event, apart from these modest prejudices of mine about the exercises—their level and their, sometimes and somewhat, improper use—I believe this is a book that every student and researcher on the subject should seriously consider having in his private library. Although making some choices—thus leaving in the shadow, as I pointed out before, some important developments—the authors have presented a mass of information, techniques and results which were previously available only in specialistic papers and therefore spread out in the literature. Also, there has been an effort to indicate, especially in the last two chapters, serious and important lines for future research. For this reason the book will be particularly useful as a guideline for research seminars and as a guidebook for all motivated graduate students and young researchers who will find in it the "real stuff", without the too-common danger of sinking into the quicksands of preliminaries.

In conclusion, my personal opinion is that this is not an easy book; rather, it is one that, despite the friendly attitude of the authors, one has to fight with. However, if you accept the challenge, you can get a lot out of it; and when you, panting, reach the end, you become aware of the fact that you liked it so much that you're tempted to go back right away to page 1 and start again.

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