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Lectures on representation theory and Knizhnik-Zamolodchikov equations, by Pavel I. Etingof, Igor B. Frenkel, and Alexander A. Kirillov, Jr., Math. Surveys and Monographs, vol. 58, Amer. Math. Soc., Providence, RI, 1998, xiv+198 pp., $49.00, ISBN 0-8218-0496-0

The equations studied in the book under review were introduced by Knizhnik and Zamolodchikov [KZ] in the early eighties as the differential equations satisfied by certain correlation functions in conformal field theory (CFT). Since then they have found applications in several areas of mathematics, including the representation theory of affine Lie algebras and quantum groups, braid groups, the topology of hyperplane complements, and the theory of knots and 3-folds. The theory of the KZ equations themselves is a generalization of the classical theory of hypergeometric functions.

1. The KZ equations

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra over $\mathbb{C}$, and fix a $\mathfrak{g}$-invariant bilinear form $(,)$ on $\mathfrak{g}$; this identifies $\mathfrak{g}^*$ with $\mathfrak{g}$ and hence corresponds to a symmetric tensor $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$. Let $V_1, \dots, V_N$ be $\mathfrak{g}$-modules (not necessarily finite dimensional), and let Ω_{ij} be Ω acting in the ith and jth factors of the tensor product $V = V_1 \otimes \cdots \otimes V_N$. The KZ equation for a V-valued function $\psi(z_1, \dots, z_N)$ is

$$\text{(KZ)} \qquad \frac{\partial \psi}{\partial z_i} = \frac{1}{\kappa} \left(\sum_{\{j|1 \le j \le N, j \ne i\}} \frac{\Omega_{ij}}{z_i - z_j} \right) \psi, \qquad i = 1, \dots, N;$$

here κ may be either a formal variable or a non-zero complex number.

The $\mathfrak{g}$-invariance of Ω has two important consequences. First, the space of solutions of (KZ) is naturally a $\mathfrak{g}$-module. This means, for example, that if V is a lowest weight module, every solution of (KZ) can be obtained from a solution with values in the space V^- of lowest weight vectors in V (or even the parts V^-_λ of V^- of a fixed weight λ). Second, (KZ) is *consistent*; i.e. if

$$\mathcal{O}_i = \frac{\partial}{\partial z_i} - \frac{1}{\kappa} \sum_{j \ne i} \frac{\Omega_{ij}}{z_i - z_j},$$

then the differential operators $\mathcal{O}_i$ and $\mathcal{O}_j$ commute for all i, j. This means that (KZ) can be interpreted as a flat connection on the trivial bundle with fibre V over the space

$$Y_N = \{(z_1, \dots, z_N) \in \mathbb{C}^N | z_i \ne z_j \text{ if } i \ne j\},$$

and solutions of (KZ) are flat sections of this bundle.

The consistency of (KZ), together with standard results in the theory of differential equations, also implies that any solution of (KZ) of the form $\psi(z_1, \dots, z_N) = z_1^{\delta_1} \dots z_N^{\delta_N} f(z_2/z_1, z_3/z_2, \dots, z_N/z_{N-1})$, where f is a formal power series and the

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δ's are constants, is automatically analytic in the region $|z_1| > |z_2| > \cdots > |z_N|$. We shall see at the end of the next section how to construct solutions of this type.

2. Affine Lie algebras and correlation functions

The affine Lie algebra $\hat{\mathfrak{g}}$ associated to $\mathfrak{g}$ is the unique non-trivial central extension, with one dimensional centre $\mathbb{C}c$, of the Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ of maps $\mathbb{C}^\times \to \mathfrak{g}$. It is well known that the theory of highest weight $\mathfrak{g}$-modules can be extended to $\hat{\mathfrak{g}}$. The irreducible highest weight $\hat{\mathfrak{g}}$-modules are parametrized by pairs (λ, k), where λ is a weight of $\mathfrak{g}$ and $k \in \mathbb{C}$. The corresponding module $L_{\lambda,k}$ is generated by a vector v that is highest weight for $\mathfrak{g}$, is annihilated by the subalgebra $\mathfrak{g} \otimes t\mathbb{C}[t]$, and is such that $c.v = kv$; the $\mathfrak{g}$-submodule of $L_{\lambda,k}$ generated by v is a copy of the irreducible $\mathfrak{g}$-module L_λ with highest weight λ. All the highest weight $\hat{\mathfrak{g}}$-modules considered from now on are assumed to be *generic*: if $\mathfrak{g} = sl_2$, so that $\lambda \in \mathbb{C}$, this means that $k \notin \mathbb{Q}\lambda + \mathbb{Q}$.

There is another family of $\hat{\mathfrak{g}}$-modules that plays an important role in this theory. If W is any $\mathfrak{g}$-module, the *evaluation module* is $W(z) = W \otimes \mathbb{C}[z, z^{-1}]$, where z is an indeterminate, with c acting trivially and

$$(x \otimes t^m).(w \otimes z^n) = (x.w) \otimes z^{m+n} \qquad (x \in \mathfrak{g}, w \in W, m, n \in \mathbb{Z}).$$

The objects of study in [KZ] were vertex operators, or primary fields. In the language of representation theory, these are certain intertwining operators between certain $\hat{\mathfrak{g}}$-modules. It is a basic result of the theory that there is a natural bijection $g \leftrightarrow \Phi^g(z)$ between $\mathfrak{g}$-module homomorphisms $g : L_{\lambda_1} \to L_{\lambda_2} \otimes W$ and $\hat{\mathfrak{g}}$-module homomorphisms $\Phi^g(z) : L_{\lambda_1,k} \to L_{\lambda_2,k} \otimes W(z)$. (We ignore here and elsewhere the necessity of using suitable completed tensor products.) To be a little more precise, we recall that $\hat{\mathfrak{g}}$, and the families of $\hat{\mathfrak{g}}$-modules we have introduced, are *graded*. Namely, $\hat{\mathfrak{g}}$ has a derivation d that annihilates c and counts the power of t in $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, the modules $L_{\lambda,k}$ have a natural grading given by the so-called Segal–Sugawara construction, and counting the power of z gives an obvious grading on the modules $W(z)$. Then, the correspondence $g \leftrightarrow \Phi^g(z)$ is characterized by the following property: if $v \in L_{\lambda_1,k}$ is of degree zero, the degree zero component of $\Phi^g(z)(v)$ is $g(v)$. However, $\Phi^g(z)$ itself is a graded map only if the grading on $W(z)$ is shifted from the obvious one: this amounts to redefining $W(z)$ to be $W \otimes z^{-\Delta}\mathbb{C}[z, z^{-1}]$, where Δ is a certain scalar depending on λ_1, λ_2, k.

Now let $L_{\lambda_1,k}, \ldots, L_{\lambda_N,k}$ be (generic) highest weight $\hat{\mathfrak{g}}$-modules; $W_{\mu_1}, \ldots, W_{\mu_{N-1}}$ lowest weight $\mathfrak{g}$-modules (not necessarily finite dimensional) with lowest weights $-\mu_1, \ldots, -\mu_{N-1}$; and $g_i : L_{\lambda_{i+1},k} \to L_{\lambda_i,k} \otimes W_{\mu_i}$ $\mathfrak{g}$-module homomorphisms ($1 \leq i \leq N-1$). We consider the map $\Psi(z_1, \ldots, z_{N-1}) : L_{\lambda_N,k} \to L_{\lambda_1,k} \otimes W_{\mu_1}(z_1) \otimes \cdots \otimes W_{\mu_{N-1}}(z_{N-1})$ given by

$$\Psi(z_1, \ldots, z_{N-1}) = (\Phi^{g_1}(z_1) \otimes \cdots \otimes 1) \ldots (\Phi^{g_{N-2}}(z_{N-2}) \otimes 1)\Phi^{g_{N-1}}(z_{N-1}).$$

It is easy to see that Ψ is a formal power series in $z_1^{-\Delta_1} \ldots z_{N-1}^{-\Delta_{N-1}}\mathbb{C}[[\frac{z_2}{z_1}, \ldots, \frac{z_{N-1}}{z_{N-2}}]]$, where the Δ_i are scalars depending on the λ's and μ's. Fix a linear form u_1 on the degree zero part of $L_{\lambda_1,k}$, and define the *correlation function* $\psi(z_1, \ldots, z_{N-1})$ with values in $V = W_{\mu_1} \otimes \cdots \otimes W_{\mu_{N-1}} \otimes L^*_{\lambda_N}$, by

$$\psi(z_1, \ldots, z_{N-1}) = \langle u_1 \, , \, \Psi(z_1, \ldots, z_{N-1})(\bullet)\rangle. \tag{1}$$

It was proved in [KZ] that the function $\psi(z_1 - z_N, z_2 - z_N, \dots, z_{N-1} - z_N)$ satisfies (KZ), with $\kappa^{-1} = k + \check{h}$ and $\check{h}$ the dual Coxeter number of $\mathfrak{g}$. Conversely, it can be shown that such correlation functions span the space of solutions of (KZ).

3. Monodromy and asymptotics

In Section 1 (from which we keep the notation) we saw that (KZ) corresponds to a flat connection on a trivial bundle over Y_N with fibre V. This means that if γ is any path in Y_N and $M_\gamma : V \to V$ is the holonomy of the connection along γ, then the *monodromy operator* M_γ depends only on the homotopy class of γ. Taking closed loops based at a point $\mathbf{z}_0 \in Y_N$, we thus get a representation of $\pi_1(Y_N, \mathbf{z}_0)$ on V. In the special case $V_1 = V_2 = \cdots = V_N$, the symmetric group $\mathfrak{S}_N$ acts on V by permuting the factors and on Y_N similarly, and we get an induced flat connection on the configuration space $Y_N/\mathfrak{S}_N$. The holonomy of this connection gives a representation of the *braid group* $\pi_1(Y_N/\mathfrak{S}_N, \mathbf{z}_0)$ on V.

It turns out that the monodromy operators can still be defined, and take on a particularly simple form, if the basepoints are replaced by "asymptotic zones". For simplicity, let $N = 3$; then (KZ) reduces to a single equation in terms of $x = (z_1 - z_2)/(z_1 - z_3)$:

$$(\mathrm{KZ}_3) \qquad \frac{\partial \psi}{\partial x} = \frac{1}{\kappa}\left(\frac{\Omega_{12}}{x} + \frac{\Omega_{23}}{x-1}\right)\psi.$$

If $v \in V$, there is a unique solution $\psi^v_{z_1 >> z_2 >> z_3}$ (resp. $\psi^v_{z_1 - z_3 >> z_1 - z_2 >> 0}$) which is asymptotic to v (in a sense we shall not make precise) as $x \to 1$ (resp. as $x \to 0$). Then we can consider the monodromy operator $M_{V_1,V_2,V_3} : V \to V$ such that $\psi^v_{z_1 - z_3 >> z_1 - z_2 >> 0} = \psi^{M(v)}_{z_1 >> z_2 >> z_3}$.

Drinfeld pointed out that there is a braided tensor category $\mathcal{C}(\hat{\mathfrak{g}}, \kappa)$ whose objects are the finite dimensional $\mathfrak{g}$-modules and in which the operators $M_{V_1,V_2,V_3} : (V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3)$ are the associativity maps; the symmetries $\sigma_{V_1,V_2} : V_1 \otimes V_2 \to V_2 \otimes V_1$ are given by $\sigma_{V_1,V_2} = \sigma e^{\pi i \Omega/\kappa}$, where σ is the switch of the factors. Moreover, he proved that, if $\kappa \in \mathbb{C}\backslash\mathbb{Q}$ or is a formal variable, $\mathcal{C}(\hat{\mathfrak{g}}, \kappa)$ is equivalent as a braided tensor category to the category $\mathcal{C}(\mathfrak{g}, q)$ of finite dimensional modules for the Drinfeld-Jimbo quantum group $U_q(\mathfrak{g})$, where $q = e^{\pi i/m\kappa}$ and $m = 1, 2$ or 3 accordingly if $\mathfrak{g}$ is of type $A/D/E$, $B/C/F$ or G. (The symmetry maps of $\mathcal{C}(\mathfrak{g}, q)$ are defined using the universal R-matrix of $U_q(\mathfrak{g})$; the associativity maps are the obvious ones.)

This last result admits a highly non-trivial modification for $\kappa \in \mathbb{Q}$, $\kappa < 0$, due to Kazhdan and Lusztig. In that case, $\mathcal{C}(\hat{\mathfrak{g}}, \kappa)$ is taken to be a certain category of $\hat{\mathfrak{g}}$-modules with the 'fusion tensor product'. Using this result, Finkelberg extended Drinfeld's result to the case $k \in \mathbb{Z}$, $k > 0$: then $\mathcal{C}(\hat{\mathfrak{g}}, \kappa)$ is replaced by the category of integrable $\hat{\mathfrak{g}}$-modules with the fusion tensor product and $\mathcal{C}(\mathfrak{g}, q)$—which is no longer semisimple, since q is a root of unity—is replaced by a certain semisimple quotient.

4. Solutions of KZ

In this section we restrict ourselves to the case $\mathfrak{g} = sl_2$ for simplicity (so that weights are just complex numbers); the solutions in this case were found by Date, Jimbo, Matsuo and Miwa. We take W_{μ_i} to be the lowest weight Verma module for $\mathfrak{g}$ with lowest weight $-\mu_i$. We observed in Section 1 that it is enough to look for

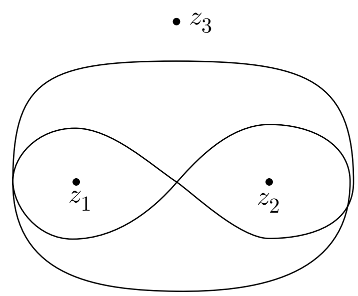

FIGURE 1

solutions of (KZ) in the finite dimensional spaces V_λ^-, where λ is necessarily of the form $\lambda = -\sum \mu_i + 2m$, with $m \in \mathbb{Z}$, $m \geq 0$.

In the simplest case $m = 0$, it is easy to see that any solution of (KZ) is of the form $\psi_0(\mathbf{z}).v$, where $\mathbf{z} = (z_1, \dots, z_N)$; v is any non-zero vector in the one–dimensional space V_λ^-; and

$$\psi_0(\mathbf{z}) = \prod_{i<j} (z_i - z_j)^{\mu_i \mu_j / 2\kappa}.$$

The case $m = 1$ is already much more complicated. If $N = 3$, it is not difficult to show that any solution of (KZ_3) can be expressed in terms of the Gauss hypergeometric function

$${}_2F_1(a, b, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} x^n,$$

where $(a)_n = a(a+1)\dots(a+n-1)$ and a, b, c are simple rational functions of μ_1, μ_2, μ_3 and κ.

For $m = 1$ and arbitrary N, the solutions can be expressed by integral formulas that generalize Euler's formula

$${}_2F_1(a, b, c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 y^{b-1}(1-y)^{c-b-1}(1-xy)^{-a} dy.$$

Let $\{e, f, h\}$ be the usual basis of sl_2, and let e_r denote e acting in the rth factor of the tensor product V. The solutions have the form

$$\Psi(\mathbf{z}) = \psi_0(\mathbf{z}) \int_C \omega_{\mathbf{z},1}.v, \tag{2}$$

where $\omega_{\mathbf{z},1}$ is the multi-valued differential form

$$\omega_{\mathbf{z},1} = \sum_{r=1}^{N} \frac{\psi_{\mathbf{z},1}(t)}{t - z_r} e_r, \quad \psi_{\mathbf{z},1}(t) = \prod_{j=1}^{N} (t - z_j)^{-\mu_j/\kappa},$$

and C can be taken to be any curve in

$$Y_{\mathbf{z},1} = \{t \in \mathbb{C} \mid t \neq z_j \text{ for all } i = 1, \dots, N\}$$

that has winding number zero about each z_j: C can still be homotopically non-trivial, as shown by the *Pochhammer loop* illustrated in Figure 1 for $N = 3$.

Finally, for m arbitrary, one replaces $\omega_{\mathbf{z},1}$ in (2) by a multi-valued differential form on

$$Y_{\mathbf{z},m} = \{\mathbf{t} = (t_1, \dots, t_m) \in \mathbb{C}^m \mid t_i \neq t_j \text{ if } i \neq j,\ t_i \neq z_r \text{ for all } i, r\}$$

of the form

$$\omega_{\mathbf{z},m} = \psi_{\mathbf{z},m}(\mathbf{t}) \times (\text{rational function of } \mathbf{t} \text{ with no poles on } Y_{\mathbf{z},m})\, dt_1 \wedge \ldots \wedge dt_m$$

and

$$\psi_{\mathbf{z},m}(\mathbf{t}) = \prod_{p<n} (t_p - t_n)^{2/\kappa} \prod_{p,j} (t_p - z_j)^{-\mu_i/\kappa}.$$

The curve C must be replaced by a suitable m-cycle on $Y_{\mathbf{z},m}$. The appropriate homology is that dual to the cohomology of the complex of holomorphic differential forms on $Y_{\mathbf{z},m}$ with coefficients in a trivial line bundle $\mathcal{L}_{KZ}$ on $Y_{\mathbf{z},m}$; the differential of this complex is given by a flat connection on this bundle which can be characterized by saying that its local sections are of the form $\psi_{\mathbf{z},m}(\mathbf{t}) \times$ (holomorphic function on $Y_{\mathbf{z},m}$). The corresponding homology groups have been computed by Aomoto.

5. Quantum KZ equation

Following the introduction of quantum groups by Drinfeld and Jimbo and the realization that CFT provides the correct geometric framework for the representation theory of affine Lie algebras (explaining, in particular, the mysterious modular properties of their characters), it was natural to look for a q-deformation of the structures of CFT. The first step in this program was taken by Frenkel and Reshetikhin [FR] with their introduction of the quantum KZ equations.

Analogues of the highest weight modules $L_{\lambda,k}$ and of the evaluation modules $W(z)$ can be defined for $U_q(\hat{\mathfrak{g}})$, and we denote them by the same symbols as in the classical case. We can then define correlation functions as in (1). But to obtain equations satisfied by these correlation functions, it is now necessary to take u_1 to be a lowest weight vector in $L^*_{\lambda_1,k}$ and evaluate ψ on a highest weight vector $u_N \in L_{\lambda_N,k}$. In other words, we now define

$$\psi(z_1, \dots, z_{N-1}) = \langle u_1\, ,\, \Psi(z_1, \dots, z_{N-1})(u_N)\rangle. \tag{3}$$

To write down the equations satisfied by ψ, we recall that, like all Drinfeld–Jimbo quantum groups, the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ has a universal R-matrix $\mathcal{R} \in U_q(\hat{\mathfrak{g}})^{\otimes 2}$ (again, a suitable completion of the tensor product should be used here); let $\mathcal{R}^{op}$ be the result of applying to $\mathcal{R}$ the switch of the factors in this tensor product. The action of $\mathcal{R}^{op}$ on a tensor product of evaluation modules $W_1(z_1) \otimes W_2(z_2)$ depends only on z_2/z_1; we write it as $R^{W_1,W_2}(z_2/z_1)$. (Actually, we should be using a 'truncated' version of $\mathcal{R}$, but we ignore this point from now on.)

Frenkel and Reshetikhin showed that the correlation function ψ defined in (3) is a solution of the system of *difference* equations

(qKZ)

$$\Psi(z_1,\dots,pz_j,\dots,z_{N-1}) = R^{W_{j-1},W_j}\left(\frac{pz_j}{z_{j-1}}\right)\dots R^{W_1,W_j}\left(\frac{pz_j}{z_1}\right)\times(q^{\lambda_1+\lambda_N+2\rho})_j$$
$$\times R^{W_j,W_{N-1}}\left(\frac{z_{N-1}}{z_j}\right)^{-1}\dots R^{W_j,W_{j+1}}\left(\frac{z_{j+1}}{z_j}\right)^{-1}\Psi(z_1,\dots,z_j,\dots,z_{N-1}).$$

In this formula, $p=q^{-2\kappa}$; R^{W_j,W_k} acts on the factors W_j and W_k in the tensor product $W_1\otimes\dots\otimes W_{N-1}$; and $q^{\lambda_1+\lambda_N+2\rho}$ means q^h, where h is the element of the Cartan subalgebra of $\mathfrak{g}$ corresponding to $\lambda_1+\lambda_N+2\rho$ under the invariant inner product on $\mathfrak{g}$, ρ being half the sum of the positive roots of $\mathfrak{g}$ and the subscript j indicating that the operator q^h acts on the jth factor W_j. Finally, we have written W_j for W_{μ_j}.

There are actually two q-analogues of (KZ); we have just described the *trigonometric form*, so-called because the matrices $R^{W_j,W_k}(z_k/z_j)$ are rational functions of $e^{\alpha z_k/z_j}$ for various constants α. There is also a *rational form*, which is related to a different quantization of $\hat{\mathfrak{g}}$ called the *double Yangian*; the corresponding R-matrices are then rational functions of z_k-z_j.

We shall not describe the solutions of (qKZ) in any detail. Suffice it to say that, in the simplest non-trivial case $\mathfrak{g}=sl_2, m=1$ (cf. Section 4), the solutions can be expressed in terms of the q-hypergeometric function

$${}_2\phi_1(q^a,q^b,q^c;q,x)=\sum_{n=0}^{\infty}\left(\prod_{j=0}^{n-1}\frac{\{a+j\}\{b+j\}}{\{c+j\}\{1+j\}}\right)x^n;$$

here, $\{a\}=(1-q^a)/(1-q)$. In the general case, one introduces appropriate discrete analogues of the vector bundles with flat connection used in Section 4.

6. Further developments

The geometric formulation of CFT is in terms of the moduli spaces of vector bundles on Riemann surfaces with marked points. The algebraic theory described here corresponds to the genus 0 case. It is thus natural to ask for 'higher genus' versions of (KZ) and (qKZ). So far, the appropriate equations have been found only in the genus 1 case: they are called the *Knizhnik–Zamolodchikov–Bernard* or *elliptic KZ* equations.

Elliptic analogues of the constructions we have described also arise when attempting to extend the equivalences of categories between representations of $\hat{\mathfrak{g}}$ and $U_q(\mathfrak{g})$ described in Section 3. On the one hand, one could replace $\hat{\mathfrak{g}}$ by $U_q(\hat{\mathfrak{g}})$: then $U_q(\mathfrak{g})$ should be replaced by a 2-parameter deformation of $\mathfrak{g}$. The appropriate object was discovered by G. Felder and is called an *elliptic quantum group* (because its R-matrices involve elliptic functions). On the other hand, one could replace $U_q(\mathfrak{g})$ by $U_q(\hat{\mathfrak{g}})$: then $\hat{\mathfrak{g}}$ should be replaced by the maps of the two-dimensional torus into $\mathfrak{g}$.

7. The book under review

This book treats all the topics we have touched on (and more) with virtually complete proofs. It begins with a mini–course on simple and affine Lie algebras and their representation theory. The authors then review the basic properties of the (KZ) equations and give an explicit construction of their solutions in terms of certain integrals. Chapter 5 studies the relation between the solutions of the (KZ) equations and vertex operators. Chapters 7 and 8 study the geometry behind the construction of solutions of the (KZ) equations. The remaining chapters are devoted to generalizing the theory to the quantum case.

The exposition simplifies many results that have so far only been accessible in difficult journal articles (the description of the homology theory of the relevant local systems is a notable example). There have, of course, been advances in the theory since the book was written. One example is the description given by Tarasov and Varchenko of the solutions of (qKZ) in terms of continuous integrals, the book including only their earlier description in terms of Jackson integrals. Nevertheless, this book will be an essential introduction for anyone wishing to enter this fascinating field at the interface between representation theory, geometry and mathematical physics.

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