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Characters and blocks of finite groups, by Gabriel Navarro, Cambridge Univ. Press, Cambridge, 1998, x+287 pp., \$39.95, ISBN 0-521-59413-4

A representation of a finite group G over a field F is said to be ordinary if F has characteristic zero and modular if F has characteristic $p > 0$. Modular representations arise naturally; any finite group of $n \times n$ matrices over a field of positive characteristic has a natural faithful modular representation of degree n. Less obviously, if G is a finite group and M and N are normal subgroups of G such that M/N is an elementary abelian p-group, then the conjugation action of G on M gives M/N the structure of a finite-dimensional, but not faithful, $GF(p)[G]$-module.

Modular representation theory (also called block theory) studies the modular representations of a finite group G and their relation to the ordinary representations of G. The subject was essentially founded by Richard Brauer (1901–1977), and Brauer remains by far the most important contributor. Modular representation theory is an exceptionally difficult subject for at least two reasons. First, the group algebra $F[G]$ fails to be semisimple whenever p divides $|G|$. Thus, in contrast to ordinary representation theory, finite-dimensional $F[G]$-modules need not be direct sums of irreducible modules. Secondly, modular representation theory seeks to connect three rather different animals: characteristic p representations, characteristic zero representations, and certain aspects of the subgroup structure of finite groups.

In order to relate the characteristic zero and characteristic p representations of G, we take F to be a sufficiently large finite field of characteristic p, so that the representation theory of G over F is the same as the representation theory of G over an algebraically closed field of characteristic p. We choose a finite extension K of the p-adic numbers whose ring of integers R maps onto F. Thus $F = R/\pi R$, where π is a prime element of the complete discrete valuation ring R, while K is the quotient field of R. We can choose F, R, and K so that the representation theory of G over K is the same as that of G over the complex numbers. Thus the set $Irr(G)$ of irreducible complex characters of G corresponds bijectively to the set of isomorphism classes of irreducible $K[G]$-modules. If $V = V_\chi$ is the irreducible $K[G]$-module with character χ, then V contains a full $R[G]$-lattice L and $L/\pi L$ is a finite-dimensional $F[G]$-module. Loosely speaking, $L/\pi L$ is the "reduction mod p" of V. The irreducible $F[G]$-composition factors of $L/\pi L$ are uniquely determined by χ.

Hence reduction mod p provides a link between the characteristic zero and characteristic p representations of G. Since most of what is needed for the ordinary representation theory of G is encoded in $Irr(G)$, one seeks an analog of $Irr(G)$ for the characteristic p representations of G. One of Brauer's first insights was to associate to an arbitrary $F[G]$-module M a complex-valued function ϕ_M, now called the Brauer character of M. Given $g \in G$, $\phi_M(g)$ is computed by lifting the eigenvalues of g on M to appropriate roots of unity in the complex numbers and then adding these roots of unity. Usually $\phi_M(g)$ is defined only for the p'-elements of G (i.e. those elements whose order is prime to p); extending the definition of ϕ_M to all of G would provide no additional information. The Brauer characters of the

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irreducible $F[G]$-modules are called the irreducible Brauer characters of G (with respect to the prime p). The number of irreducible Brauer characters of G equals the number of isomorphism classes of irreducible $F[G]$-modules, which equals the number of conjugacy classes of p'-elements of G.

If $\chi \in Irr(G)$, then the Brauer character of the reduction $\bmod p$ of V_χ is a nonnegative integral linear combination of the irreducible Brauer characters of G. Thus $\chi^0 = \sum_\phi d_{\chi\phi}\phi$, where χ^0 is the restriction of χ to the p'-elements of G, ϕ ranges over the irreducible Brauer characters of G, and the $d_{\chi\phi}$ are non-negative integers. The $d_{\chi\phi}$ are called the decomposition numbers of G; they are the numerical invariants that most directly link the characteristic zero and characteristic p representations of G.

Having defined the Brauer characters and decomposition numbers with respect to the prime p, we turn next to the p-blocks of G. In $Z(F[G])$, the center of the modular group algebra, we can decompose the identity as $1 = e_1 + \cdots + e_m$, where the e_i are mutually orthogonal primitive idempotents. Thus $F[G] = B_1 \oplus \cdots \oplus B_m$, where the $B_i = e_i F[G]$ are mutually annihilating indecomposable algebras, called the block algebras of $F[G]$. Since the discrete valuation ring R is complete, the e_i can be lifted to mutually orthogonal idempotents $\tilde{e}_1, \ldots, \tilde{e}_m$ of $Z(R[G])$. Then $R[G] = \widetilde{B}_1 \oplus \cdots \oplus \widetilde{B}_m$, where the $\widetilde{B}_i = \tilde{e}_i R[G]$ are mutually annihilating indecomposable ideals of $R[G]$.

We can view the $\tilde{e}_i$ as central idempotents of $K[G]$. Then if V_χ is as above, there is a unique index i such that $\tilde{e}_i$ does not annihilate V_χ. We then say, with slight abuse of language, that χ belongs to the block B_i. We denote by $Irr(B_i)$ the set of all $\chi \in Irr(G)$ that belong to B_i. Similarly, if M_ϕ is an irreducible $F[G]$-module affording the irreducible Brauer character ϕ of G, there is a unique index i such that e_i does not annihilate M_ϕ; we then say that ϕ and M_ϕ belong to the block B_i. It is easy to see that the decomposition number $d_{\chi\phi}$ is zero unless χ and ϕ belong to the same block. Thus the decomposition matrix $(d_{\chi\phi})$ of G is the direct sum of the decomposition matrices of the individual blocks.

The concepts of Brauer character, decomposition number, and block flow rather easily from the goal of relating the characteristic zero and characteristic p representations of G. The same cannot be said, however, of Brauer's crucial concept of the defect group of a block. The defect group of a p-block B is a p-subgroup D of G, which is defined up to conjugacy in G. For example, the defect groups of the block that contains the trivial character are the p-Sylow subgroups of G. One important and easily stated result says that if $\chi \in Irr(B)$ and $g \in G$, then $\chi(g) = 0$ unless the p-part of g is contained in a defect group D of B. (If g is an element of an arbitrary finite group G, then g can be written uniquely as a commuting product of a p-element and a p'-element of G; these factors are respectively called the p-part and the p'-part of g.) Furthermore, if $\chi \in Irr(B)$, then the p-part of the index $|G : D|$ divides the degree $\chi(1)$ of χ.

At a deeper level, "the complexity of the group-theoretic structure of D measures the complexity of the structure of the block algebra B of $F[G]$," to quote from Robinson's authoritative and readable recent survey [10]. For example, when $D = 1$, the block algebra is isomorphic to the field F and $Irr(B)$ consists of a single character χ, which vanishes off the p'-elements of G. Furthermore, the degree of χ is divisible by the p-part of $|G|$, and the reduction $\bmod p$ of V_χ is an irreducible $F[G]$-module.

Brauer's three main theorems, like the several alternative definitions of the defect group of a block, are too technical to state here. The main theorems relate the p-block B of G to p-blocks of certain subgroups of G. These subgroups are "p-local"; usually the p-local subgroups of G are defined to be the normalizers of the nonidentity p-subgroups of G, but for our purposes we will broaden the definition to include subgroups of such normalizers. The first main theorem, for example, sets up a bijection between the blocks of G with defect group D and the blocks of $N_G(D)$ with defect group D. The second main theorem, especially important for the applications of block theory, goes a long way toward expressing the values of the ordinary irreducible characters in B at elements of order divisible by p, in terms of the irreducible Brauer characters of certain p-local subgroups of G.

Much research in block theory has been guided by a few major conjectures. Before stating some of these, we recall that if B is a p-block of G with defect group D and $\chi \in Irr(B)$, then the p-part of $|G : D|$ divides $\chi(1)$. Thus if $|D| = p^d$ and p^a is the highest power of p dividing $|G|$, then we can write the exact power of p dividing $\chi(1)$ as p^{a-d+h}, with $h \geq 0$. We call h the height of χ and $d-h$ the defect of χ. We let $k(B) = |Irr(B)|$, and for $0 \leq e \leq d$ we denote by $k_e(B)$ the number of $\chi \in Irr(B)$ that have defect e.

Brauer himself made a number of major conjectures, most of which are listed in [3]. Two of the most important are the height zero conjecture and the $k(B)$ conjecture. With B and D as above, the former conjecture predicts that D is abelian if and only if every $\chi \in Irr(B)$ has height zero, and the latter conjecture predicts that $k(B) \leq |D|$. Compared to some of the later conjectures, Brauer's have the appearance of modest empirical observations. This is characteristic of Brauer, who combined extraordinary insight and technical ability with unpretentious, relatively down-to-earth methods.

Brauer's major conjectures remain unproven. In the last fifteen years, more attention has been given to the conjectures of Alperin, Dade, and Broué. Perhaps the most compelling of these is Dade's, which is a descendent of Alperin's. One part of Dade's conjecture predicts an explicit formula for the block invariant $k_e(B)$ defined above as an explicit alternating sum of the block invariants $k_e(b)$, as b ranges over an appropriate set of blocks of certain p-local subgroups of G. This would be part of an extremely precise solution to the central problem of relating the block invariants of G to those of p-local subgroups of G.

It should be added that conceptual proofs (i.e. proofs not using the classification of finite simple groups) of any of the conjectures above seem very remote. It is more likely that some of these conjectures could be proved using the simple group classification together with information about the various families of simple or nearly simple groups. At least one of the conjectures above, however, is very difficult even for solvable groups.

Navarro's book is intended primarily for graduate students or others who are familiar with ordinary character theory but who have had no previous exposure to modular representation theory. His book is a natural sequel to I. M. Isaacs' widely praised book on ordinary character theory [8]. Like Isaacs' book, Navarro's features good organization, clear and detailed proofs, and good exercises. Like Isaacs, Navarro includes a number of interesting results that are not essential to the main development.

Following Brauer's original methods, Navarro makes little use of representations or modules in characteristic p. The proofs of most results involve ordinary characters, Brauer characters, and other related complex-valued functions. In conjunction with these complex-valued functions, substantial use is made of certain subalgebras of the characteristic zero and characteristic p group algebras of G and its subgroups. Navarro's expository style, however, is clearly that of Isaacs rather than Brauer's.

A module-theoretic approach to the subject was developed in the 1960's by J. A. Green. In Green's theory, the important objects are the indecomposable modules for the group algebras $F[G]$ and $R[G]$, where $F = R/\pi R$ as above. This approach leads to summands of restricted and induced modules, endomorphism rings of modules, and homological and categorical ideas.

Many results have both character-theoretic and module-theoretic proofs. For example, Green proved using modules that the defect group of a block is always an intersection of two p-Sylow subgroups (the actual result is sharper than this). Navarro's book contains a character-theoretic proof by Robinson of this result. Brauer proved that if $\chi \in Irr(B)$ and $g \in G$, then $\chi(g) = 0$ unless the p-part of g lies in a defect group of B, as we mentioned above. Navarro's book contains a character-theoretic proof of this result. An alternative module-theoretic proof uses Green's notion of the vertex of an indecomposable module.

The module-theoretic point of view is well represented by Alperin's book [1], written at about the same level as Navarro's. Concise and authoritative, Alperin's book is narrower in scope than Navarro's in that it covers only the characteristic p theory. Thus the reader who is new to the subject can choose between two enthusiastic introductions with starkly contrasting points of view. For those who desire a more neutral and comprehensive text, a good choice would be Nagao and Tsushima [9]. The most comprehensive book on the subject is Feit's treatise [6], which is, however, not suitable for beginners.

Block theory has much to offer to ordinary character theory; this is, and should be, a major theme of Navarro's book. I think Navarro could have done even more to develop this theme. For example, some of Brauer's deepest and most important results, dating from the early 1940's and proved in Navarro's book, concern groups whose order is divisible by a prime to (exactly) the first power. Navarro gives an example to show that these results are sufficient to determine the degrees of the ordinary irreducible characters of a hypothetical simple group of order 7920. However, one might go further by showing that Brauer's results go a long way toward determining the entire ordinary character table of such a group.

On a more theoretical note, a major tool in proving the deeper results of ordinary character theory is the use of bijective correspondences between the ordinary irreducible characters of a group G and those of an appropriate subgroup H. One such correspondence is the Glauberman correspondence, in which G is acted on by a coprime automorphism group A and $H = C_G(A)$, the fixed point subgroup of A. As Alperin noticed, the Glauberman correspondence is essentially a consequence of Brauer's first main theorem. This connection is given in Navarro's book, but is relegated to the exercises in such a way that an unsophisticated reader might fail to recognize its significance.

Another way in which block theory enriches ordinary character theory is through the specialization of some of the major conjectures in block theory to p-solvable groups. (A group is called p-solvable if each of its composition factors is either a p-group or a p'-group.) This leads to problems in ordinary character theory which are often important and interesting and sometimes, as in the case of Brauer's $k(B)$ conjecture, extremely difficult. Navarro mentions some of this in his book.

The classification of finite simple groups, completed about 1980, has shifted the emphasis from abstract finite group theory to the study of specific families of groups: for example, the finite classical groups, the other families of finite groups of Lie type, the symmetric and alternating groups, and the sporadic simple groups. In view of this shift in emphasis and the inherent difficulty of block theory, it is not surprising that the last twenty years have seen more progress in the modular representation theory of specific families of finite groups than in the general theory for abstract finite groups.

As late as the 1960's little was known about the modular representation theory of specific families of groups (the symmetric groups being an exception), while great progress was being made in the general theory. In addition to the development of Green's theory mentioned above, Brauer's results on groups of order divisible by a prime to the first power (also noted above) were generalized by Dade [5] to arbitrary blocks with cyclic defect group. This required a key observation by J. G. Thompson involving Green's theory.

Block theory also played a limited but significant role in the classification of finite simple groups. One such application is Glauberman's Z^*-Theorem, which implies that if T is a 2-Sylow subgroup of a finite simple group G and $t \in T$ has order 2, then T contains an element which is G-conjugate to but not equal to t. The proof of this theorem, which appears in Navarro's book, depends on Brauer's main theorems. Brauer's difficult technical work in [2] and related papers are another major use of block theory in the simple group classification.

Since the 1970's the ordinary representation theory of the finite groups of Lie type has been completely transformed by the elaborate and sophisticated theory of Deligne and Lusztig; see e.g. [4]. Deligne-Lusztig theory has also led to major advances in the modular representation theory of the finite groups of Lie type, which include most of the finite simple groups. An influential contribution in this direction has been Fong and Srinivasan's work on the finite general linear and unitary groups [7].

One might ask what goals are reasonable in the modular representation theory of a specific infinite family of finite groups. Among other things, one would hope to parametrize the ordinary irreducible and irreducible Brauer characters of the groups in this family and then use these parametrizations to determine the distribution of characters into blocks and to find the decomposition matrices of these blocks. Even the first step of parametrizing the ordinary irreducible characters may be quite difficult. For the groups of Lie type, Deligne-Lusztig theory provides a parametrization of the ordinary irreducible characters, but a conjectured parametrization of the irreducible Brauer characters when p is not the natural characteristic of the group has yet to be proved. One goal that is not reasonable for infinite families is to compute the values of the irreducible Brauer characters; very little has been done in this direction even for the symmetric groups.

Much recent work in block theory has involved verifying some of the major conjectures of the theory for some specific families of finite groups (and also for some individual groups). This of course unites the theoretical side of block theory with the work being done on specific groups. Although the proofs of the major conjectures are not imminent, verifying them for some specific groups has been a complex and worthwhile undertaking.

In summary, modular representation theory is an exceptionally difficult subject, still greatly influenced by the achievements and ideas of Richard Brauer. Navarro's book is a strong pedagogical work and a worthy sequel to Isaacs' book on ordinary character theory.

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