BOOK REVIEWS

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Non–vanishing of L–functions and applications, by M. Ram Murty and V. Kumar Murty, Progress in Mathematics, Birkhäuser Verlag, Basel, Boston, London, 1997, 196 pp., $52.00, ISBN 3-7643-5801-7

The theory of L–functions and analytic number theory goes back to 1837 when Dirichlet introduced the L–series [D, chapter 4] (Dirichlet series)

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s},$$

(where χ is a character of the cyclic group $(\mathbb{Z}/q\mathbb{Z})^*$ for a positive integer q) to prove that there are infinitely many primes congruent to $a \pmod{q}$ provided a is coprime to q. The key step in Dirichlet's proof was showing that

$$L(1,\chi) \neq 0.$$

Riemann in 1860 [D, chapter 8] pointed out the deep connection between the distribution of primes and the zeroes of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

Hadamard and de la Valleé Poussin in 1899 [D, chapter 13] were the first to prove the prime number theorem (that the number of primes less than x is asymptotic to $x/\log x$) by showing that $\zeta(s)$ does not vanish for $Re(s) \geq 1$. The key idea was to utilize the simple trigonometric identity

$$3 + 4\cos(\theta) + \cos(2\theta) \geq 0$$

applied to

$$Re\Big(\log \zeta(\sigma + it)\Big) = \sum_p \sum_{m=1}^{\infty} \frac{\cos\left(t \log p^m\right)}{m p^{m\sigma}}$$

to obtain

$$\Big|\zeta^3(\sigma)\, \zeta^4(\sigma + it)\, \zeta(\sigma + 2it)\Big| \geq 1,$$

which is a contradiction as $\sigma \to 1$ if $\zeta(1 + it) = 0$, since a fourth order zero at $1 + it$ kills the triple pole at $\sigma = 1$.

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The above classical results provide the motivation for the current monograph which focuses on non–vanishing results on classical Dirichlet L–functions, Artin L–functions, and L–functions associated to modular forms. A number of non–vanishing techniques are studied, foremost of which are generalizations of the Hadamard and de la Valleé Poussin idea and methods of averaging to show that a large percentage of L–functions in a family do not vanish at a particular point.

The book begins by generalizing the idea of Hadamard and de la Valleé Poussin and applying it to give a sketch of a proof of the prime number theorem for arithmetic progressions. Then zeta functions of number fields are introduced as well as Hecke L–functions for number fields, and the prime number theorem is generalized to these situations.

Next, Artin L–functions are introduced. Let L/K denote a finite extension of number fields with Galois group G. Let $\mathfrak{p}$ be a prime of K and $\mathfrak{q}$ be a prime of L with $\mathfrak{q}|\mathfrak{p}$. Define O_K, O_L to be the ring of integers of K, L, respectively. Set

$$D_{\mathfrak{q}} = \{\sigma \in G \mid \sigma\mathfrak{q} = \mathfrak{q}\}$$

to be the decomposition group of $\mathfrak{q}$, and

$$I_{\mathfrak{q}} = \{\sigma \in D_{\mathfrak{q}} \mid \sigma x \equiv x \pmod{\mathfrak{q}}, \forall x \in O_K\}$$

to be the inertia group of $\mathfrak{q}$. Then $D_{\mathfrak{q}} = \mathrm{Gal}\,(L_{\mathfrak{q}}/K_{\mathfrak{p}})$ where $L_{\mathfrak{q}}$ (resp. $K_{\mathfrak{q}}$) is the completion of L (resp. K) at $\mathfrak{q}$ (resp. $\mathfrak{p}$), and there exists a group homomorphism

$$D_{\mathfrak{q}} \longrightarrow \mathrm{Gal}\left(O_L/\mathfrak{q}\Big/O_K/\mathfrak{p}\right)$$

with kernel $I_{\mathfrak{q}}$. The quotient group $D_{\mathfrak{q}}/I_{\mathfrak{q}}$ is a finite cyclic group generated by the Frobenius element $\sigma_{\mathfrak{q}}$ which satisfies

$$\sigma_{\mathfrak{q}}(x) \equiv x^{N\mathfrak{p}} \pmod{\mathfrak{q}}, \quad \forall x \in O_L,$$

and N denotes the norm. If we choose another prime $\mathfrak{q}'$ above $\mathfrak{p}$, then $I_{\mathfrak{q}'}$ and $D_{\mathfrak{q}'}$ are conjugates of $I_{\mathfrak{q}}$ and $D_{\mathfrak{q}}$.

Let ρ be a representation of G with character χ. Note that for any $\sigma \in G$, the trace (i.e. the character χ) and the determinant of $\rho(\sigma)$ depend only on the conjugacy class in which σ lies. Let V be the underlying complex vector space on which ρ acts. Let $\mathfrak{p}$ be a prime of K and $\mathfrak{q}$ any prime of L above $\mathfrak{p}$. Then we may restrict this action to the decomposition group $D_{\mathfrak{q}}$, and we see that the quotient $D_{\mathfrak{q}}/I_{\mathfrak{q}}$ acts on the subspace $V^{I_{\mathfrak{q}}}$ of V on which $I_{\mathfrak{q}}$ acts trivially. For $Re(s) > 1$, we define the Artin L–function

$$L(s,\chi,K) = \prod_{\mathfrak{p}} \det\Big(I - \big(\rho(\sigma_{\mathfrak{q}}) \mid V^{I_{\mathfrak{q}}}\big)\ N\mathfrak{p}^{-s}\Big)^{-1}$$

where the product goes over primes $\mathfrak{p}$ of K. The most important open problem on these L–functions is Artin's conjecture which states that $L(s,\chi,K)$ has a meromorphic continuation to the whole complex s–plane with at most one pole at $s = 1$ of order equal to the multiplicity of the trivial representation in ρ.

The book goes on to prove the Aramata–Brauer Theorem, which states that the quotient $\zeta_L(s)/\zeta_K(s)$ of Dedekind zeta functions is entire provided L/K is a Galois extension. Some generalizations of this result are also obtained for finite non–Galois extensions L/K. Non–vanishing results for Artin L–functions are then established. It is shown that there is a zero free region just slightly to the left of the line $Re(s) = 1$. This result is then applied to prove the Chebotarev density Theorem and obtain results on the least prime in a conjugacy class.

The third chapter introduces general formalism due to Serre relating analytic continuation of L–functions of compact groups and uniform distribution. A very general version (due to P. Deligne) of the Hadamard and de la Valleé Poussin technique is used to prove non–vanishing for $Re(s) \geq 1$.

The book continues with a brief overview of the theory of modular forms and L–functions of modular forms. Let Γ be a subgroup of finite index of the modular group $SL(2, \mathbb{Z})$. A modular form of weight k is a holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ (where $\mathfrak{h}$ denotes the upper half–plane) satisfying

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. If $\begin{pmatrix} 1 & M \\ 0 & 1 \end{pmatrix} \in \Gamma$ for some positive integer $M \in \mathbb{Z}$, then $f(z+M) = f(z)$ and f will have a Fourier expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n \, e^{\frac{2\pi i n z}{M}}.$$

Associated to f there is an L–function

$$L_f(s) = \sum_{n=1}^{\infty} a_n \, n^{-s}$$

which will have an analytic continuation and satisfy a functional equation. Hecke showed that $L_f(s)$ will have an Euler product if f is an eigenfunction of certain operators called Hecke operators. In this case, there are well known conjectures on the distribution of the coefficients a_p for primes p (Sato–Tate conjecture). Non–vanishing results of L–functions associated to modular forms (and their symmetric power analogues) are discussed. The chapter concludes with proofs of some theorems of R. Murty [RM] on the oscillations of the Fourier coefficients.

It was conjectured by S. Chowla that if χ is a real quadratic Dirichlet character, then the classical Dirichlet L–function $L(s, \chi)$ never vanishes at $s = \frac{1}{2}$. This conjecture is still unproven and is expected to be true for all Dirichlet characters. It was shown by Balasubramanian and K. Murty [BKM] that if q is a sufficiently large prime, then $L\left(\frac{1}{2}, \chi\right) \neq 0$ for at least $\geq 0.04q$ of the Dirichlet characters χ (mod q). The method of proof of this result is based on the study of averages

$$\sum_{\chi (\text{mod } q)} L\left(\frac{1}{2}, \chi\right) M_z\left(\frac{1}{2}, \chi\right)$$

where $M_z(s, \chi)$ is a Dirichlet polynomial of length z which *mollifies* the L–function.

Let $f(z)$ be a holomorphic cusp form of weight 2 and character ϵ for the congruence subgroup $\Gamma_0(N)$. Assume that f is an eigenfunction of all the Hecke operators. Let

$$L(s, f) = \sum_{n=1}^{\infty} a(n) n^{-s}$$

denote the L–function associated to f. For any integer D let χ_D denote the quadratic Dirichlet character $\pmod D$ and define

$$L_D(s,f) = \sum_{n=1}^{\infty} a(n)\chi_D(n)\, n^{-s}$$

to be the twisted L–function. K. Murty proved that there exist infinitely many fundamental discriminants D such that $L_D(1,f) \neq 0$, and this is the main theme of Chapter 6. The proof of this result is obtained by several difficult technical estimates on weighted character sums. This type of result is the missing ingredient in Kolyvagin's [Ko] sensational proof of the finiteness of the Tate–Shafarevich group for modular elliptic curves and was obtained independently by Bump–Friedberg–Hoffstein (see [BFH]) by different methods. Curiously, the connection to the Tate–Shafarevich group is barely mentioned.

The book concludes with a chapter on Selberg's conjectures concerning Dirichlet series with Euler products and functional equations (called Selberg's class). These conjectures imply that no L–function in the Selberg class can vanish on the line $Re(s) = 1$. It is a folklore conjecture that the Selberg class coincides with the class of automorphic L–functions.

The Murty–Murty book gives an excellent presentation of Artin L–functions with very elegant proofs of their basic properties. The applications of the non–vanishing results to the least prime in a conjugacy class, the relation between Artin's conjecture and the Selberg conjectures, and the very general prime number theorem of Deligne are beautifully presented. This material is not accessible in any other single place. It should be remarked that half the book is spent on giving detailed proofs of theorems of the Murtys on average values of Dirichlet L–functions and twisted modular L–functions using character sum estimates in a manner which is entirely out of sync with the spirit of the earlier chapters, where key ideas and conjectures are presented in a leisurely easy-to-read manner.

The main weakness of the book is what is missing. The work of Waldspurger [W] (made explicit by Kohnen–Zagier [KZ]) is barely touched on. Kohnen [K] shows that starting with a holomorphic newform of f weight k, there is an associated modular form of weight $\frac{k+1}{2}$ whose q^{th} Fourier coefficient is a constant times a power of q times $\sqrt{L_f(1,\chi)}$ where χ is a real quadratic character $\pmod q$. This is the most natural way to get mean value theorems on L–functions twisted by quadratic characters. Goldfeld and Hoffstein [GH] have obtained mean value theorems for real Dirichlet L–functions using the theory of metaplectic Eisenstein series. These results have been generalized to number fields and higher rank groups by Bump–Friedberg–Hoffstein (see [BFH]). Another method for obtaining non–vanishing results which has not been touched on is due to Rohrlich [R], who makes use of Galois averages. Finally, important results and applications due to Iwaniec, Sarnak, their students, and many others (see [I]) have been omitted.

To conclude, the first four chapters and the last chapter provide an excellent introduction to the modern analytic theory of L–functions. The most important families of L–functions (Dirichlet L-functions, Artin L–functions, Modular Form L–functions, L–functions of the Selberg class), their main properties, and non–vanishing results are presented in a leisurely, easy-to-read style, getting quickly to the key ideas and heart of the matter. The almost 100 pages of proof for the two mean value theorems seems a bit out of place in relation to the earlier chapters, and other methods and techniques for obtaining such results have been omitted.

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