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Topological methods in hydrodynamics, by V. I. Arnold and B. A. Khesin, Springer, New York, 1998, xv + 374 pp., \$59.95, ISBN 0-387-94947-X

1. Introduction

Let us consider the motion of an ideal incompressible homogeneous fluid in a fluid container M, an n-dimensional compact oriented C^∞ Riemannian manifold with C^∞ boundary ∂M and metric g. A mathematical model for such a fluid is provided by the Euler equations of ideal hydrodynamics

$$\begin{gathered} \partial_t u(t,x) + \nabla_u u(t,x) = -\text{grad } p(t,x), \\ \text{div } u = 0, \quad u(0) = u_0, \\ g(u,n) = 0 \text{ on } \partial M, \end{gathered} \tag{1}$$

where the unknown variable $u(x,t)$ is a time dependent vector field on M representing the spatial or Eulerian velocity, and the pressure function $p(x,t)$ is completely determined by u via the Hodge projection. We use the notation ∂_t to denote the partial time derivative, ∇ to denote the Levi-Civita covariant derivative associated to g on M, and n to denote the outward normal vector field on ∂M.

Let $\mathcal{D}_\mu$ denote the Lie group of C^∞ diffeomorphisms of M which preserve the volume form μ and leave the boundary ∂M invariant. Group multiplication is given by composition of maps. For any $\eta \in \mathcal{D}_\mu$, the tangent space $T_\eta \mathcal{D}_\mu$ consists of the smooth maps $v : M \to TM$ which cover η and satisfy $\text{div}(v \circ \eta^{-1}) = 0$ and $g(v \circ \eta^{-1}, n) = 0$. Let $\langle \cdot, \cdot \rangle$ denote the right invariant Riemannian metric on $\mathcal{D}_\mu$ given at the identity $e \in \mathcal{D}_\mu$ by the L^2 inner-product

$$\langle u, v \rangle_e = \int_M g(u(x), v(x))\mu, \quad \forall u, v \in T_e \mathcal{D}_\mu. \tag{2}$$

The vector space $T_e\mathcal{D}_\mu$ is the Lie algebra of C^∞ divergence-free vector fields on M which are tangential to ∂M, with Lie bracket $[u,v] = \nabla_v u - \nabla_u v$. In his seminal paper [1], Arnold proved that a curve $\eta(t)$ in $\mathcal{D}_\mu$ is a geodesic of $\langle \cdot, \cdot \rangle$ if and only if the projection of $\dot\eta(t)$ onto the Lie algebra $T_e\mathcal{D}_\mu$, given by $u(t) = \dot\eta(t) \circ \eta(t)^{-1}$, is a solution of the Euler equations (1). With this result, Arnold was able to transfer the problem of studying the evolution equation (1) to the problem of finding geodesics of $\langle \cdot, \cdot \rangle$ on $\mathcal{D}_\mu$ and, in particular, to the study of the geometry of the volume-preserving diffeomorphism group. Thus began the modern theory of topological or geometric hydrodynamics. Since then, a large number of researchers have contributed greatly to this remarkably vast research area, and Arnold and Khesin's book is a wonderfully rich survey of the progress made over the last thirty years. It is an invaluable reference to researchers already working in the area of topological hydrodynamics and an indispensable learning tool for those wishing to enter the subject.

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2. Lagrangian and Eulerian fluid motion

Symmetry reduction is a fundamental tool for reducing the number of equations governing the dynamics of a physical system. In the simplest setting, the configuration space of a dynamical system is a Lie group G (or more generally a topological group), and the dynamics are governed by a Lagrangian function $L : TG \to \mathbb{R}$ which is either left or right invariant with respect to the lifted action of G on TG. For concreteness, let us consider the right invariant case; then, there is a natural symmetry reduction given by $TG \mapsto TG/G \cong \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G with right Lie bracket $[\cdot,\cdot]_R$. Letting R_g denote right translation by $g \in G$, the above reduction is given by $(g, \dot{g}) \mapsto TR_{g^{-1}}\dot{g}$. The problem, then, is to formulate the equations of motion for the reduced variables on the reduced phase space $\mathfrak{g}$. The solution is supplied by what is now called the Euler-Poincaré reduction theorem, which we state in the context of right invariant systems following the development of Marsden and Scheurle [17].

Theorem 2.1 (Euler-Poincaré)**.** *Let G be a topological group which admits smooth manifold structure with smooth right translation, and let $L : TG \to \mathbb{R}$ be a right invariant Lagrangian. Let $\mathfrak{g}$ denote the fiber T_eG, and let $l : \mathfrak{g} \to \mathbb{R}$ be the restriction of L to $\mathfrak{g}$. For a curve $\eta(t)$ in G, let $u(t) = TR_{\eta(t)^{-1}}\dot{\eta}(t)$. Then the following are equivalent:*

a) *the curve $\eta(t)$ satisfies the Euler-Lagrange equations on G;*

b) *the curve $\eta(t)$ is an extremum of the action function*

$$S(\eta) = \int L(\eta(t), \dot{\eta}(t))dt,$$

for variations $\delta\eta$ with fixed endpoints;

c) *the curve $u(t)$ solves the Euler-Poincaré equations on $\mathfrak{g}$*

$$\frac{d}{dt}\frac{\delta l}{\delta u} = -ad_u^*\frac{\delta l}{\delta u},$$

where the coadjoint action ad_u^* is defined by

$$\langle ad_u^* v, w\rangle = \langle v, [u, w]_R\rangle,$$

for u, v, w in $\mathfrak{g}$, and where $\langle\cdot,\cdot\rangle$ is the metric on $\mathfrak{g}$ and $[\cdot,\cdot]_R$ is the right bracket;

d) *the curve $u(t)$ is an extremum of the reduced action function*

$$s(u) = \int l(u(t))dt,$$

for variations of the form

$$\delta u = \dot{w} + [w, u],$$

where $w = TR_{\eta^{-1}}\delta\eta$ vanishes at the endpoints.

Interested readers should also see [14].

In the case of incompressible hydrodynamics, where $G = \mathcal{D}_\mu$ and $l = \int_M g(u, u)\mu$, the functional derivative $\delta l/\delta u = u$, so that geodesic flow satisfies $\partial_t u = -\mathrm{ad}_u^* u$ on $\mathfrak{g}$. It is a simple matter to verify that $\mathrm{ad}_u^* u = \nabla_u u + \mathrm{grad}\, p$; hence, geodesics of (2) are solutions of (1). The symmetry group is the massive particle relabeling symmetry of hydrodynamics, arising from the action of $\mathcal{D}_\mu$ on $T\mathcal{D}_\mu$. Presented as reduction on the tangent bundle $T\mathcal{D}_\mu$, this is Euler-Poincaré reduction, while the identical reduction on the cotangent bundle side (where the Lagrangian is replaced by the Hamiltonian) is termed Lie-Poisson reduction. In terms of classical fluid mechanics terminology, the unreduced motion of the fluid particles on $T\mathcal{D}_\mu$ is termed the Lagrangian or material representation, while evolution of the spatial velocity field on the reduced space $T_e\mathcal{D}_\mu$ is termed the Eulerian representation. Chapter I of *Topological Methods in Hydrodynamics* presents a thorough development of this reduction process as well as the Lie-Poisson reduction of $T^*\mathcal{D}_\mu$ onto $T_e^*\mathcal{D}_\mu$, where the Euler equations, expressed in terms of 1-forms, are given by $\partial_t u + \pounds_{u^\sharp} u = -d\tilde{p}$, where $u^\sharp$ is the vector field associated to u and $\pounds$ denotes the Lie derivative (see [18] as well).

3. The geometry of $\mathcal{D}_\mu$

The entirety of Chapter IV is devoted to the geometry of $\mathcal{D}_\mu$ and begins with Arnold's famous curvature computation that first appeared in [1] and later in Appendix 2 of [2]. As an application of his new methodology, Arnold tackled the problem of *idealized* weather prediction by studying the Lagrangian stability of fluid particles. Because solutions of the Euler equations are geodesics of (2) on $\mathcal{D}_\mu$, one may study the linearized stability problem by considering solutions of the Jacobi equation along a given geodesic curve; it follows that the sectional curvature K of the metric (2) on $\mathcal{D}_\mu$ completley determines the growth rate of initial errors. Arnold made a few simplifying assumptions, approximating the surface of the Earth by $\mathbb{T}^2$ and the trade-wind current by a sinusoidal stream-function, and proceeded to compute $K < 0$ at the identity for all possible perturbations. Consequently, exponential growth of initial errors results, prohibiting accuracy in long-term weather prediction. Other authors have since obtained similar results for the volume-preserving diffeomorphism group of $\mathbb{S}^2$ and other compact Riemannian manifolds (see the authors' extensive bibliography).

In the case that M is a bounded subset of $\mathbb{R}^n$, $\mathcal{D}_\mu$ is naturally isometrically embedded into the vector space $L^2(M, \mathbb{R}^n)$, and this embedding has led to some remarkable analytic results, some of which are described in a section written by Shnirelman. Before describing these results and their connection with the traditional PDE approach to (1), it must be noted that Arnold and Khesin's presentation is strictly formal, in that they do not specify the topology of $\mathcal{D}_\mu$, instead referring the reader to the 1970 paper of Ebin and Marsden [7], wherein C^∞ differentiable structure is supplied to a slightly enlarged class of volume-preserving diffeomorphisms.

Let H^s denote the Hilbert space whose elements and their first s distributional derivatives are in L^2, and consider the set of Hilbert class volume-preserving diffeomorphisms of M given by

$$\mathcal{D}_\mu^s = \{\eta \in H^s \mid \eta \text{ is bijective}, \eta^{-1} \in H^s, \eta^*(\mu) = \mu\}.$$

The well-known theorem of Ebin and Marsden [7] states that whenever $s > (n/2) + 1$, $\mathcal{D}_\mu^s$ is a C^∞ topological group with smooth right multiplication. Because left multiplication and inversion are only continuous (as maps of $\mathcal{D}_\mu^s$ into $\mathcal{D}_\mu^s$), the set $\mathcal{D}_\mu^s$ is not a Lie group. In fact, the group exponential map does not even cover a neighborhood of the identity; nevertheless, this smooth topological group in many ways behaves like a Lie group because of the remarkable geodesic properties established in [7]. Namely, Ebin and Marsden proved that (1) could be reexpressed as the ODE

$$\begin{gathered}\ddot{\eta}(t) = S(\eta, \dot{\eta}), \\ \eta(0) = e,\ \dot{\eta}(0) = u_0,\end{gathered} \tag{3}$$

where S is in $C^\infty(T\mathcal{D}^s_\mu, T^2\mathcal{D}^s_\mu)$. The vector field S is the geodesic spray of $\langle\cdot,\cdot\rangle$ on $\mathcal{D}^s_\mu$ and is a smooth zeroth-order differential operator; consequently, the fundamental theorem of ordinary differential equations on Hilbert manifolds immediately gives local existence and uniqueness of C^∞ geodesics that depend smoothly on the initial data.[1] Recall that these are geodesics of a weak metric, that is, a metric which induces a topology on $\mathcal{D}^s_\mu$ which is weaker than the original H^s topology, $s > n/2+1$, and that in general, weak metrics *do not* generate geodesic flow.

A corollary to the Ebin-Marsden result is that in a sufficiently small neighborhood of the identity, any two fluid configurations can be joined by an energy minimizing curve, but Shnirelman has proven the surprising result that when $M = [0,1]^3$, this local minimization is not globally attainable; namely, there does not exist a minimal geodesic connecting any two fluid configurations (see [23]). Hence, the calculus of variations fails in the large on $\mathcal{D}^s_\mu$, but, by relaxing the regularity and invertability requirements of fluid configurations, it is possible to construct a generalized flow (a curve in this weakened space of fluid configurations) which is indeed globally energy minimizing. Both Shnirelman and Brenier [3], [23] developed this notion, but Brenier's latest result [3] is truly spectacular, as he constructs Young-measure-valued solutions to the (barely) convex global minimization problem on the space of Lebesgue measure preserving maps of $[0,1]^3$, which are both Eulerian and Lagrangian in nature and sharpen the Eulerian Young-measure-valued solutions of DiPerna and Majda [6] – averaging Brenier's solution with respect to the Lagrangian variable recovers the DiPerna and Majda result. With the exception of the most recent results, the book provides an extremely readable presentation of these ideas (previously only available in a collection of difficult-to-read papers) all in one concise chapter.

4. Other hydrodynamical-type systems

Amazingly, many other hydrodynamical systems arise from symmetry reduction on either a central extension or a semi-direct product of certain diffeomorphism groups.

For example, the equations of magneto-hydrodynamics (MHD) are the geodesic equations on the semi-direct product of $\mathcal{D}_\mu$ with $\Omega^1/d\Omega^0$ (Ω^k denotes the vector space of differential k-forms on M), with respect to the right invariant metric given at the identity by $\int_M [g(u,v) + g(B,C)]\mu$ where $(u,B),(v,C)$ are in the Lie algebra $T_e\mathcal{D}_\mu \times (\Omega^1/d\Omega^0)$. See Holm and Kuperschmidt [10] to see this development using Clebsch variables and Marsden, Ratiu and Weinstein [16] for the geometric picture using semidirect product theory.

The compressible Euler equations are obtained through Lie-Poisson reduction on the semi-direct product of $\mathcal{D}$ with $C^\infty(M)$ (see Marsden [13]) and are governed

[1] This result, in turn, gives sharp local well-posedness of classical solutions to the Euler equations on compact Riemannian manifolds with boundary. It should be noted that the classical PDE approach to the Euler equations yields only C^0 solution curves in H^s and C^0 dependence on initial data, so that with respect to analytic considerations, it is better to work over the entire tangent bundle $T\mathcal{D}^s_\mu$ as opposed to the single fiber $T_e\mathcal{D}^s_\mu$.

by the reduced Hamiltonian

$$H(u,\rho) = -\int_M \left(\frac{1}{2}\rho g(v,v) + \phi(\rho)\right)\mu,$$

where ρ is the density of the fluid, and $(d/d\rho)\phi(\rho) = h(\rho)$, the pressure function. Of course, inhomogeneous incompressible fluids can be handled in a similar fashion.

When $M = \mathbb{S}^1$, the KdV equation arises as geodesic flow with respect to the right invariant L^2 metric on the Bott-Virasoro group (in addition to the text under review, see [14] as well for a nice discussion of this central extension); this is the set $\mathcal{D} \oplus \mathbb{R}$ with multiplication law

$$(\phi(x), a) \circ (\psi(x), b) = \left(\psi(\psi(x)), a + b + \int_{\mathbb{S}^1} \log(\phi \circ \psi(x))' d \log \psi'(x)\right).$$

More recently, the completely integrable shallow water equation

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + \kappa u_{xxx} = 0$$

extensively studied by Camassa and Holm [4] (see also [8]) has been formally shown in [20] to arise as geodesic flow on the Bott-Virasoro group with right invariant metric given at the identity by the H^1 inner-product $\int_{\mathbb{S}^1}(u^2 + u_x^2)dx$. In the most interesting case when $\kappa = 0$, this equation admits peaked soliton solutions with infinite slope [4], thus modeling the breaking wave phenomenon while maintaining complete integrability. With $\kappa = 0$, this equation is a geodesic of the weak H^1 right invariant metric on the diffeomorphism group of the circle $\mathcal{D}^s(\mathbb{S}^1)$ for $s > 3/2$ (the Hilbert class diffeomorphisms). It has been shown that C^∞ geodesics of this weak metric exist due to the fact that the geodesic spray is smooth (see [21], [22]), and thus well-posedness of this PDE is given for initial data in $H^s(\mathbb{S}^1)$ for $s > 3/2$. Classical hyperbolic PDE techniques have not yielded this sharp result (see the beautiful results of [5]).

The authors present many other examples as well; of particular interest is the amazing fact that the completely integrable filament equation for the time evolution $\gamma(t,x)$ of the initial curve $\gamma(0,x)$, $x \in \mathbb{S}^1$ given by

$$\frac{\partial \gamma}{\partial t} = k(t,x)\frac{\partial \gamma}{\partial x} \times \frac{\partial^2 \gamma}{\partial x^2}, \quad k = \text{ curvature of } \gamma,$$

is the Hamiltonian function with respect to the Marsden-Weinstein symplectic structure on the space of knots (see also [18] and [16]). This, in turn, is related to Chorin's vortex blob method in three-dimensions for integrating the Euler equations and the evolution of vortex filaments. In a new development, Chorin's vortex method (with a particular choice of blob) is a geodesic on $\mathcal{D}^s_\mu$ with respect to a new right invariant metric given at the identity by

$$\langle u, v\rangle + \frac{\alpha^2}{2}\langle \pounds_u g, \pounds_v g\rangle, \quad \alpha > 0, \tag{4}$$

where $\langle\cdot,\cdot\rangle$ is the L^2 inner-product (see [22]). The resulting PDE is given on the interior of M by

$$\begin{gathered} \partial_t(1-\alpha^2\triangle_r)u + \nabla_u(1-\alpha^2\triangle_r)u - \alpha^2(\nabla u)^t \cdot \triangle_r u = -\text{grad } p, \\ \text{div } u = 0, \quad u(0) = u_0, \quad \triangle_r = -(d\delta + \delta d) + 2\text{Ric}. \end{gathered} \tag{5}$$

Because of the complicated nature of the mixed spacetime partial differential operators appearing in this PDE, it is a surprising fact that when treated in terms of the Lagrangian variables $(\eta, \dot\eta)$, this PDE is in fact an ODE governed by a vector field with no derivative loss. In other words, the geodesic spray of (4) is a C^∞ bundle map, taking an H^s class bundle into an H^s bundle for $s > (n/2) + 1$ (see [15] and [22]). This PDE is also the equation for a second-grade non-Newtonian fluid, and this connection may shed light on the intriguing relationship between filament behavior in turbulent regimes and polymer flow, which has been addressed in the turbulence literature. Finally, (5) is also the averaged Euler or Euler-α equations considered by Holm, Marsden, and Ratiu [11] and analyzed by Marsden, Ratiu, and Shkoller [15].

It should be emphasized that the Arnold approach to fluids has been extremely influential in both the ocean dynamics and the plasma physics community, both for stability theory, which we mention next (see [16]), as well as for studying Arnold-type dynamics on the group of *symplectic* diffeomorphisms (a nice discussion of this can be found in Marsden and Weinstein [19]).

5. Further topological methods

There are a great number of additional topics covered by Arnold and Khesin that have yet to be mentioned. Among these is the extremely important and interesting notion of stability of stationary solutions to the Euler equations and, more generally, to Lie-Poisson systems on the duals of Lie algebras, known as the Arnold or energy-Casimir method. This method, as well as its generalization, the energy-momentum method (presented in a short section written by Marsden), is thoroughly reviewed in Chapter II; the interested reader should also see [12] and [14]. Finally, there is a superb chapter on the topological properties of magnetic and vorticity fields, wherein it is shown that the Helicity invariant for divergence free vector fields ξ, defined in a simply connected domain $M \subset \mathbb{R}^3$ by $\mathcal{H}(\xi) = \langle \xi, \mathrm{curl}^{-1}\xi \rangle$, is the average self-linking of ξ and is related to the self-linking of knots in a magnetic field associated with ξ. Freedman's solution of the Sakharov-Zeldovich problem of energy-minimization of the unknotted magnetic field and the lower bound of the $L^{\frac{3}{2}}$ norm of ξ by the asymptotic crossing-number is given a very clear presentation (see [9]), and there is a short section on asymptotic holonomy wherein the Jones-Witten invariant as well as the Chern-Simons functional are discussed. The authors write in the preface that "some statements in this book may be new even for the experts," and this certainly seems to be the case.

In short, there is an enormous wealth of content provided by the authors, much of it not to be found in any other single source. Certainly, all topological and geometric techniques for studying hydrodynamics are discussed, so the title is unquestionably appropriate. This book, and its extensive bibliography, should serve as a tremendous reference for all researchers in the field and certainly belongs on the bookshelf next to other classic mathematical references.

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