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Discriminants, resultants and multidimensional determinants, by I. M. Gelfand, M. M. Kapranov, and A.V. Zelevinsky, Birkhäuser, Boston, 1994, vii + 523 pp., $82.00, ISBN 0 817 63660 9

1. Introduction

Being a book lover, often as a result of the pleasant pastime of browsing through library shelves on some loosely busy afternoon, I end up buying many books. The selection is usually made on the basis of the choice of the topics presented, but sometimes also on the basis of the ratio: fame of the author/price of the book. Too often afterwards, when I start to read the book (not necessarily from the beginning), I can be very disappointed, and the book goes back to the shelf to remain beautifully untouched. A quite different fate occurred to the book by Gelfand, Kapranov and Zelevinsky (GKZ for short), and I will try here to explain the reasons for this (at the same time trying to justify the long delay of the present review by the fact that I wanted to read the book and not just have a quick look at it).

This book has several peculiar virtues, the first one being leading us pleasantly along a beautiful road which on the one hand comes from far away in the past, on the other hand projects us into the future. The theories illustrated in the book are indeed deeply rooted in the mathematics of last century (1800!), yet of extreme current interest. To explain this seeming contradiction, I cannot refrain from quoting Abhyankar's motto "Eliminate the eliminators of elimination theory!" (cf. [Abh]).

As is probably known, elimination theory (which occupies a central position in the book) is the theory which, given a set of polynomial equations $g_j(x, y) = 0$ where $x = (x_1, ..x_n), y = (y_1, ..y_m)$, allows us to find equations $r_i(x) = 0$ which are satisfied, for a given x, if and only if there exists a y such that (x, y) is a common solution of the equations $g_j(x, y) = 0$. It is called elimination theory simply because, in logical terms, we have eliminated the predicate "exists a y such that"; in practical terms, we have eliminated the variables y, whence, continuing to apply the procedure, we can reduce the study of systems of equations in several variables to systems of equations of higher degree but each involving one single variable. All of this was started on not very firm theoretical foundations by people like Leibniz, Cramer and Euler in the 1700's. In the second half of the century elimination theory was founded on the basis of direct algebraic manipulations by Bezout (cf. [Bez1], [Bez2], and [BrN] for a quite detailed historical account). Later on, in the next century, the investigations focused on several interactions, e.g. with the theory of algebraic invariants, and produced a vast amount of explicit and complicated calculations. Here, the most eccentric exponent of the school seems to have been Paul Gordan, about whom Hermann Weyl wrote ([Wey2]): "There exist papers by him where twenty pages of formulas are not interrupted by a single text word; it is told that in all his papers he himself wrote the formulas only, the text

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being added by his friends." In fact, we owe to G. Kerschensteiner the editing of his lectures ([GoK1],[GoK2]).

In our century, the emphasis was set more on abstract ideas than on algorithms and recipes; indeed Andre Weil wrote in his book ([Wei]) the famous sentence "The device that follows, which, it may be hoped, finally eliminates from algebraic geometry the last traces of elimination theory...." For many people of my generation the introduction to mathematics was so general and abstract (since we first had to face all sorts of topological spaces before we would be confronted with simple mathematical objects like conics or quadrics) that we were looking for something concrete to understand before daring to launch ourselves again to abstract concepts. The direction was reversing, and Abhyankar's motto marks a revision of the tendency, very much motivated by the computer revolution and a new feeling that perhaps formidable calculations could no longer be completely out of reach.

Why should one go back to the past? The answer is simple. In the past mathematicians dealt with problems coming from other sciences. For instance, the theory of discriminants is intimately related to our vision schemes. What we see best of objects is their boundary, or more precisely a projection of their boundary. That is, while the surface boundary Σ of an object is 2-dimensional, the visible boundary Γ is the 1-dimensional geometrical object parametrizing the family of rays issuing from our eyes and touching the surface boundary Σ of the given object. Thus the visible boundary is the contact curve Γ consisting of the points P for which the line-rays joining P and our eye O are tangent to the surface Σ in P. If we assume we set up projective coordinates where our eyes (or the sun's light) is at infinity on the z-axis and our surface boundary Σ is described by a polynomial equation $F(x, y, z) = 0$, we are looking for the projection on the (x, y)-plane of the contact curve Γ defined by the pair of equations $F(x, y, z) = 0, \frac{dF}{dz}(x, y, z) = 0$. This is precisely a very particular case of the elimination problem we considered above; we have two polynomial equations, and we would like to eliminate the variable z from them, thus obtaining the equation $\Delta(x, y) = 0$ of the plane picture (projection) of the visual boundary (as is well known, the singularities of Δ allow us to recognize the 'shapes'). The desired polynomial equation $\Delta(x, y)$ is here given by the so-called discriminant of F with respect to the variable z (we shall discuss this concept more amply later).

Another elementary issue where discriminants pop up is for instance in the notion of "envelopes" in the theory of ordinary differential equations (cf. [SC]). The simplest way to analyse this concept is to abstractly consider everything exactly as before, except that we view the equation $F(x, y, z) = 0$ as a family of plane curves C_z parametrized by the parameter z. Then the points (x_0, y_0) where $\Delta(x, y)$ does not vanish are, by Dini's implicit function theorem, such that if (x_0, y_0) belongs to the curve C_{z_0}, then for all points in a neighbourhood of (x_0, y_0) we can write the parameter z as a function $g(x, y)$: this means that through each point there is exactly one curve C_z if the parameter z is rather near to z_0 (or, in still other words, our curves are locally given as the level sets of the function $g(x, y)$). This translates into a unicity result for first order ordinary differential equations. Assume in fact that we replace our variable z throughout by the variable $p = dy/dx$; then from the "implicit" differential equation $F(x, y, dy/dx) = 0$ we obtain locally an "explicit" differential equation of the first order, i.e., $dy/dx = g(x, y)$. And our discriminant curve of equation $\Delta(x, y)$ contains the singular solutions of our differential equation. There are thus several related concrete issues (focal loci, caustics, bifurcation phenomena for P.D.E.'s) which illustrate the central role of discriminants, and this explains why the first word in the title of the book is just 'discriminants'.

On the other hand, in the traditional mathematical education of the present time, the three clue words appearing in the title of the book follow each other (at least in the one dimensional case) in the opposite order: namely, first determinants, then resultants and finally discriminants. In all these cases we have a polynomial in several indeterminates (e.g., the entries of a square matrix) which has the special property that only relatively few monomials enter into its expression.

Problem 1. Roughly speaking, the main problems for a "special" polynomial such as a determinant, a hyperdeterminant, a discriminant, resultant, or generalized resultant are first to understand the geometry of its zero locus; second to understand the set of the occurring monomials; finally to determine, if possible, the corresponding coefficients by means of elegant formulae.

The case of the determinant of a square matrix $detA =$

$$det \begin{pmatrix} a_{1,1} & a_{1,2} & .... & a_{1,n} \\ a_{2,1} & a_{2,2} & .... & a_{2,n} \\ ... & ... & .... & .... \\ a_{n,1} & a_{n,2} & ... & a_{n,n} \end{pmatrix} \qquad (*)$$

is completely solved thanks to the well known Leibniz formula

$$det(A) = \sum_{\sigma \in \mathcal{S}_n} \epsilon(\sigma)(\Pi_{i=1}^n a_{i,\sigma(i)}).$$

Here, the coefficient $\epsilon(\sigma)$ is the signature of the permutation σ, equal to $+1$ or -1, so in this case the problem of the determination of the coefficients is explicitly solved. The case of the resultant is more complicated: recall that usually the resultant of two polynomials $f(x) = a_0 + a_1 x + ..a_n x^n$, $g(x) = b_0 + b_1 x + ...b_m x^m$ or, better, of the two homogeneous polynomials

$$F(x_0, x_1) = x_0^n f(x_1/x_0) = a_0 x_0^n + a_1 x_0 x_1^{n-1} + ...a_n x_1^n$$

and

$$G(x_0, x_1) = x_0^m g(x_1/x_0) = b_0 x_0^m + b_1 x_0 x_1^{m-1} + ...b_m x_1^m$$

is introduced via the Sylvester determinant, the determinant of the following Sylvester matrix $A_{n,m}(f, g) =$

$$\begin{pmatrix} 0 & ... & 0 & a_0 & a_1 & ... & a_n \\ ... & ... & ... & ... & ... & ... & 0 \\ 0 & a_0 & a_1 & ... & a_n & 0 & ... \\ a_0 & a_1 & ... & a_n & 0 & ... & 0 \\ b_0 & b_1 & ... & .. & b_m & 0 & ... \\ 0 & ... & ... & ... & ... & ... & 0 \\ ... & 0 & b_0 & b_1 & ... & ... & b_m \end{pmatrix}. \qquad (*)$$

In fact, cf. [Coo], one can more generally define $r_k(f, g)$ as the determinant of the square minor A_k of order $(m + n + 2 - 2k)$ obtained by deleting the first and last $(k - 1)$ rows and columns of $A_{n,m}(f, g)$. Recall that the crucial property of these polynomial expressions in the respective coefficients of f, resp. g, is given by the following classical

Theorem 1. *The homogeneous polynomials* $F(x_0, x_1) = x_0^n f(x_1/x_0), G(x_0, x_1) = x_0^m g(x_1/x_0)$ *have a common divisor of degree* $\geq k$ *if and only if* $r_1(f, g) = ... = r_k(f, g) = 0$.

The reason why the theorem holds can be easily explained, since the rows of the matrix express the polynomials $x^{m-1}f, ..., xf, f, g, xg, ...x^{n-1}g$ in the standard basis for polynomials. Therefore, the vanishing of the resultant $r(f, g) = r_1(f, g)$ expresses the linear dependence of the above vectors. This amounts to saying that their least common multiple has lower degree than $(n + m)$= expected degree of fg, and this is clearly equivalent to saying that the greatest common divisor of F, G is not a constant. One problem with the resultant is, however, that it is not so easy to calculate: a very nice short-cut reducing it to the determinant of a $(n \times n)$ matrix is presented in the lovely chapter 12 of the book, entitled "An overview of classical formulas". The trick consists in first reducing oneself to the case where $f(0) = a_0 \neq 0$ (this is rather trivial) and then considering the rational function g/f and its Taylor development at the origin (MacLaurin development): $g(x)/f(x) = r(x) = r_0 + r_1 x + r_2 x^2 + ...$ The condition that f and g have a common multiple of degree smaller than $n + m$ can be rephrased as:

- there exist polynomials P of degree at most $n - 1$ and Q of degree at most $m - 1$ such that Pg is congruent to fQ (mod x^{m+n})
  and in turn as:
- there exists a polynomial P of degree at most $n - 1$ such that the power series Pg/f is congruent mod (x^{m+n}) to a polynomial Q of degree at most $m - 1$.

Thus we get the condition that the n columns (of length n) corresponding to the coefficients of $x^n, \dots x^{m+n-1}$ in the functions (power series) $g/f, xg/f, \dots x^{n-1}g/f$ yield a matrix R whose determinant must vanish. From this we obtain the equation $detR =$

$$det \begin{pmatrix} r_m & r_{m+1} & .... & r_{m+n-1} \\ r_{m-1} & r_m & .... & r_{m+n-2} \\ ... & ... & .... & ... \\ r_0 & r_1 & .... & r_{n-1} \end{pmatrix} = 0. \qquad (*)$$

The matrix R and its determinant (the so-called Schur polynomial; cf. [Fu1], [ACGH]) are well known in the theory of determinantal varieties (cf. for these also the classical [Ro]) and are used in the so-called Porteous' formula yielding the locus where a map of vector bundles drops rank. From the computational complexity point of view this formula yields a major improvement because it reduces drastically the number of operations which are necessary. There arises naturally the question whether one can do better. A false impression that one could do much better comes from the so-called interpolation formula: this is a beautiful formula for the resultant, which is often taken as a definition. If we write the polynomials f, g in terms of their roots, $f(x) = a_n \Pi_{i=1}^n (x - \alpha_i)$, $g(x) = b_m \Pi_{j=1}^m (x - \beta_j)$, the interpolation formula expresses the resultant as follows: $r(f, g) = \epsilon(n, m)\ a_n^m\ b_m^n\ (\Pi_{i=1}^n \Pi_{j=1}^m (\alpha_i - \beta_j)) = \epsilon'(n, m)\ a_n^m\ (\Pi_{i=1}^n g(\alpha_i))$. Again, here $\epsilon(n, m), \epsilon'(n, m)$ are equal to just $+1$ or -1, but the clearly elegant formulae require a nonallowable operation, that is, root extraction!

The interpolation formula makes quite visible, however, the reason why the resultant vanishes when f, g have a common root and can be particularly appreciated in the case of the discriminant. The discriminant of a polynomial $f(x)$ as above is defined as the resultant of $f(x)$ and its derivative, and then one obtains the formula:

$$\delta_f := Res_x(f(x), f'(x)) = (-1)^{\epsilon(n)} a_n^{2n-1} (\Pi_{i<j}(\alpha_i - \alpha_j)^2).$$

The last formula is important in the Galois theory of field extensions where, though, one is concerned with monic polynomials and can thus dispense with the term a_n^{2n-1}. Writing now the discriminant δ_f as a polynomial in the coefficients of $f(x)$, we still get a certain order of divisibility by the leading term a_n: getting rid of this factor, one obtains the more familiar discriminant $\Delta(f)$. The concrete calculation of $\Delta(f)$ for polynomials of degree n presents interesting coefficients which tend to be products of the first n numbers with rather large exponents:

$$\begin{aligned} \Delta(a_0 + a_1 x + a_2 x^2) &= 4a_0 a_2 - a_1^2 \\ \Delta(a_0 + a_1 x + a_2 x^2 + a_3 x^3) &= 27 a_0^2 a_3^2 + 4 a_0 a_2^3 + 4 a_1^3 a_3 - a_1^2 a_2^2 - 18 a_0 a_1 a_2 a_3 \end{aligned}$$

([GKZ], pages 405-406).

The interesting part of the story is that resultants and discriminants are not parents and children, but more cousins of each other, as taught to us by Cayley. In fact, given two polynomials $f(x)$ and $g(x)$ as before, we can consider the polynomial in three variables $P(x, y_0, y_1) = y_0 f(x) + y_1 g(x)$ and observe that the statement "exists x such that $f(x) = g(x)$" is equivalent to the statement "exists (x, y_0, y_1) where P and its partial derivatives vanish". This follows right away since the three partial derivatives of P are $f(x), g(x), y_0 f'(x) + y_1 g'(x)$, and the third is a linear equation in the two unknowns y_0, y_1; thus the necessary condition that $f(x), g(x)$ vanish is also a sufficient condition. The morals of this simple-minded example are two. The first one, which is not new, is the old moral that it is always better to consider homogeneous polynomials: if we had considered just the two-variable polynomial $Q(x, y) = f(x) + y g(x)$, we would have had the problem that the equation $f'(x) + y g'(x)$ does not have a solution y when $g'(x) = 0$, but $f(x) \neq 0$. The second moral is deeper and is just that we are naturally led to consider an analogous notion of discriminant for polynomials of several variables. However, this generalization brings us directly in medias res with the theory consistently and progressively developed in the chapters of the book.

2. The abstract set up

A far-reaching generalization of the notion of discriminant is obtained through a rather old geometric idea, namely the theory of projective duality, which, after precursor results by Pascal and Desargues in 1600, was established around 1818-1827 by Poncelet, Gergonne and Moebius. (Poncelet called his theory "Theorie generale des polaires reciproques", while the word "duality" seems to have been first introduced by Gergonne: cf. the second edition of [Po], especially the pages 359-396 of the "Section supplementaire" for an interesting account of the more than 40 years of priority fights between the two French geometers.)

Here, the final outcome is the concept of the dual variety $X^\vee$ (almost always a hypersurface) of a projective variety X (cf. [Wa1], [Wa2]). To explain it, we need to recall how the concept of duality in linear algebra arose geometrically. If V is a vector space, we associate to it the projective space $\mathbf{P}(V)$ whose points are the 1-dimensional linear subspaces of V, so that the points in the projective space $\mathbf{P}(V^\vee)$

correspond to codimension 1 subspaces of V, the so-called "hyperplanes". Now, if X is an algebraic variety in $\mathbf{P}(V)$—i.e., X is defined by finitely many polynomial equations—then also the set $X^\vee$, defined as the closure of the set given by the hyperplanes tangent to X in a smooth point of X, forms an algebraic variety $X^\vee$ in $\mathbf{P}(V^\vee)$. In 'almost' all cases $X^\vee$ is a hypersurface, which means that $X^\vee$ is defined by a single polynomial equation; classifying the exceptions to this behaviour is an intriguing, still open problem. Let us only recall that for surfaces in 3-space the only exceptions are cones and tangential developable surfaces (note that cylinders are the same thing as cones in projective geometry!).

The connection with discriminants comes in directly when we take as X the n-th Veronese embedding X^1_n of a projective line $\mathbf{P}^1$, classically called the rational normal curve of order n. X^1_n is the image of the map $v_n : \mathbf{P}^1 \to \mathbf{P}^n$ given by all monomials of degree n, $v_n(x_0, x_1) = (x_0^n, x_0^{n-1}x_1, ...x_1^n)$, and the vector space of hyperplane sections of X^1_n is just the vector space of homogeneous polynomials of degree n in (x_0, x_1). Saying that a hyperplane is tangent means exactly that the corresponding polynomial does not have n distinct roots; whence $a_0y_0 + a_1y_1 + ...a_ny_n = 0$ is a tangent hyperplane if and only if the discriminant $\Delta(a_0, a_1, ...a_n)$ of the polynomial $a_0x_0^n + a_1x_0x_1^{n-1} + \dots a_nx_1^n$ does vanish. At this point it is clear that we can extend the notion of discriminants immediately to polynomials $P(x_0, x_1, \dots x_m)$ of several variables simply by considering the n-th Veronese embedding $X^m_{n^m}$ of $\mathbf{P}^m$.

We come here to an interesting historical and pedagogical point: to show that $X^\vee$ is also an algebraic variety, one needs to apply a more general form of multivariable elimination theory, the theory of u-resultants (cf. [vdW]). Thus we seem to enter into a vicious circle: resultants yield elimination theory among whose many byproducts are the dual varieties, and in turn dual varieties produce generalized resultants. Indeed, the modern abstract methods provide more elegant (on top of the cited [Wei], cf. also [Mum]) but nonconstructive proofs that in projective geometry elimination of variables leads to polynomial equalities only (that is, no inequalities, as would occur in the following example: seeking the set of values for x such that there is y so that the equation $xy = 1$ has solutions, we get the obvious necessary and sufficient condition that $x \neq 0$). On the other hand, in the present book we find one of the first systematic and combined expositions of the theory of dual varieties and of the theory of Chow varieties. This is given in part I, under the title "General discriminants and resultants"; although the topic is classic (indeed the book also contains a brief but informative set of historical notes), we find here a lot of new material, and many presentations are new, for instance the Lagrangian point of view in the exposition of the theory of Chow varieties.

At this point I need to explain what a Chow variety is and what relations this notion has with the theory of resultants. In general, if one takes r polynomials $f_1(x_1, ..x_n), .., f_r(x_1, ..x_n)$ of degree d in n variables and asks whether they have a common root, one can surely first of all reduce to the case where they are linearly independent. Afterwards, one can consider them as giving a subspace U of dimension r of the space of polynomials of degree d and slightly reformulate the problem as follows : in the so-called Grassmann space parametrizing all such linear subspaces U of dimension r, what are the equations that must be satisfied in order that all the polynomials in U have a common root?

We want once more to emphasize that our original problem involves equalities and inequalities, so we are only considering the first part of the question. And, in order to answer the first question, it is easier to consider, instead of a single r-tuple, the totality of such subspaces U. What we have gained is that in this way we also allow roots "at infinity"; namely we consider instead of the original polynomials their homogenization $F_1(x_0, x_1, ...x_n), ..., F_r(x_0, x_1, ...x_n)$, and we can phrase our question then geometrically as follows : given the Veronese image X_d^n of $\mathbf{P}^n$ (under the map whose coordinates are given by all monomials of degree d in $(x_0, x_1, ...x_n)$), when does the codimension r subspace $U^\vee$ dual to U intersect X_d^n? At this point dimension theory tells us that if $r \leq n$, then no equations have to be satisfied by the coefficients of $F_1, ..., F_r$ in order for this intersection to be nonempty; whereas the case $r \geq n+1$ can be reduced to the case $r = n+1$ by the following trick: $F_1, ..., F_r$ have a common (nontrivial) solution if and only if every (n+1) linear combination of $F_1, F_2..., F_r$ has a common root. (Viewed geometrically, this means that if $U^\vee$ and X_d^n do not intersect, then we can find a linear subspace of codimension $n+1$, containing $U^\vee$, such that it does not intersect X_d^n.) Moreover, since in the case $r = n+1$ one gets a single polynomial equation, the algebraic counterpart of this trick is that for $r \geq n+1$ this single equation in the coefficients of those linear combinations is indeed a polynomial in the indeterminates $\lambda_{i,j}$ parametrizing the (n+1) linear combinations $\sum_j \lambda_{i,j} F_j$, $(i = 1, ...n+1)$. Since we want this polynomial to vanish for each choice of the linear combinations, and since its vanishing is equivalent to the vanishing of its coefficients as a polynomial in the indeterminates $\lambda_{i,j}$, we finally obtain in this way a system of polynomial equations in the coefficients of the original polynomials $F_1, ..., F_r$.

Therefore we have just seen how the problem of resultants is related to the following question: given a projective variety X of dimension n, find a polynomial equation for the codimension $n+1$ subspaces $U^\vee$ which have a nonempty intersection with X. Such a polynomial exists, is unique up to constants, and, in an appropriate projective space of polynomials, yields the so-called Chow form of X. Its importance in projective geometry is that the knowledge of the Chow form completely determines our variety X. Finally, the Chow variety is the set of all Chow forms of the projective varieties of a given dimension n and a given degree m, and, as the words suggest, it is indeed an algebraic variety for which explicit polynomial equations can be written. The name Chow variety is used in spite of the fact that the idea goes back to Cayley (1860), just because Cayley limited himself to considering the case of curves in a three dimensional space. The theory was later generalized by Bertini, van der Waerden and Chow, and recently Green and Morrison generalized Cayley's method of writing equations for the Chow variety.

One of the purported aims of the book is to show how much deeper the contributions of Cayley were, and indeed how his papers on the theory of elimination and on combinatorics were deeply related to each other. The authors emphasize the revolutionary idea of the determinant of a complex which goes back to Cayley. This notion has been several times rediscovered and extended by several authors—namely by Reidemeister, Franz and De Rham, later by Whitehead and Milnor—and is recently at the centre of current research, through the notions of determinants of families of differential operators, introduced by Quillen and Ray and Singer, and of determinants of the cohomology, introduced by Mumford-Knudsen and later by Deligne and Faltings. Since this is a central idea, we will try to explain it, presenting it in a rather special case: let $0 \to V_1 \to V_2 \to \ldots \to V_r \to V_{r+1} \to 0$ be a complex of linear maps $a_i : V_i \to V_{i+1}$ between vector spaces, each endowed with a fixed basis (one can easily generalize all the notions to the case of free modules over a ring). Saying that we have a complex amounts to requiring that each composition $a_{i+1} \circ a_i = 0$. If we assume moreover that the Euler Poincarè characteristic $\sum_j (-1)^j \dim(V_j)$ of the complex equals zero, and if we let A_i be the matrix of the linear map a_i, then there is a polynomial $\mathrm{Det}(A_1, ...A_r)$ which vanishes precisely when the complex is not exact, i.e., when for some i there holds $\mathrm{Im}(a_i) \subset_{\neq} ker(a_{i+1})$. Here again, abstract algebraic geometry shows without great difficulty that the above polynomial exists: what is more interesting is that (cf. theorem 14, page 485) Cayley gave in 1848 formulae expressing this polynomial as a rational function whose denominator and numerator are explicit products of determinants of certain minors of the matrices A_i.

The reader might wonder now what this may have to do with resultants and discriminants: this is not easy to see unless one has some familiarity with the commutative algebra concept of the Koszul complex. This complex is exact as soon as a series of elements $F_1, ...F_r$ forms a regular sequence (geometrically this notion simply means that each new equation makes the dimension of the solution set drop down by 1): in our particular case, if $r = n + 1$, our polynomials yield a regular sequence exactly if they have no common zeroes on X as a subset of the given projective space (i.e., the only solution is given by setting $x_i = 0\ \forall i$). Therefore, the Chow form of our variety X is obtained by calculating the determinant of the complex associated to the subspace U spanned by $F_1, \dots F_{n+1}$. In turn, the method of Cayley also allows us in principle to write down explicitly the equation $\Delta_X(f)$ of the dual variety $X^\vee$ (it vanishes when the hyperplane corresponding to f is tangent to X). Thus, for instance, in the case of the Veronese embedding of projective space yielding the discriminant of a general polynomial, we could think that we know everything (at least in low degree and low number of variables); but if we then ask a computer to write the discriminant, we obtain in response a formula filling several pages. Well, then our eyes simply are not broad enough to read the formula; therefore our next main purpose will be to rewrite it, if possible, in such a way that we can interpret it or just apply it to special classes of polynomials for which it might become simpler. Two typical instances where the discriminants become simpler are:

- Presence of symmetry (cf. for a broader philosophical issue, [Wey1]): the polynomials are invariant under the action of a group, often a finite one, and we can apply representation theoretic considerations.
- Case of sparse polynomials (cf. e.g. [Stu]): many monomials have coefficients equal to zero.

It is clear from what we have said that resultants and discriminants already give examples of sparse polynomials, but to illustrate the concept we may choose a much simpler basic example of a set of sparse polynomials. Among all homogeneous polynomials of degree n, let us take those which are multilinear: this example will allow us to introduce one of the basic newer concepts treated in the book, namely the concept of toric varieties.

3. Toric varieties

If we consider in the affine plane two general conics whose equations are general quadratic polynomials, then they intersect in 4 distinct points. If instead we consider two general bilinear equations (i.e., they involve only the monomials $1, x, y, xy$), then we shall see that the intersection points of the corresponding hyperbolae are just two. One geometric way to see this is to observe that those hyperbolae have parallel asymptotes; whence, in the projective plane, two of the 4 intersection points stay fixed at infinity (they correspond to the two fixed directions of the asymptotes). A better way to see this is that the map given by the four monomials has as image a one sheeted hyperboloid in 3-space, whence it gives a compactification of the affine plane which is not the projective plane but the product of two projective lines $\mathbf{P}^1 \times \mathbf{P}^1$. Two such hyperbolae in the plane correspond to plane sections of the hyperboloid; whence the two intersection points of two such hyperbolae correspond to the two intersection points of the hyperboloid with a line in $\mathbf{P}^3$.

A similar situation occurs for instance with the famous eight quadratic equations which occur in the robotic arm problem: the number of solutions of these equations in eight unknowns is not 256 as one would immediately conjecture, since the equations, although complicated, are easily seen to be bilinear with respect to a splitting of the eight variables into two sets. A similar argument as above easily shows that, for such general bilinear equations, the expected number of solutions should indeed be 64 (we refer to [SoWa] for a recent survey on these kinds of problems). The above examples are based on the fact that to different spaces of polynomials correspond different compactifications of the affine space (in the first example, $\mathbf{P}^1 \times \mathbf{P}^1$ instead of $\mathbf{P}^2$). The theory of toric varieties goes all the way through and considers a set A of Laurent monomials in $k-1$ variables, i.e., monomials having also possibly negative coefficients: whence results a map f_A of $H := (\mathbf{C}^*)^{k-1}$ in a projective space of dimension $|A|-1$, and in case f_A is an embedding, the closure X_A of the image is called a projective toric variety.

In general a toric variety is a variety X with an action of the algebraic group $H := (\mathbf{C}^*)^{k-1}$ and having an open dense orbit isomorphic to H. Expositions of the theory are given in [Fu2], [O]; here a quicker and easier presentation is given by emphasizing the role of projective toric varieties (in savant language, these are the pairs of a toric variety and of an equivariant projective embedding). The essential feature of toric varieties is the synergy of methods from the geometric theory of convex bodies with algebro-geometric methods (indeed, the interaction, as often in mathematics, takes place in both directions; cf. [CSh]) and of methods from representation theory (here, one views the Laurent monomials as characters of the algebraic torus H).

In the above situation we consider A as a subset of $\mathbf{Z}^{k-1}$, and we consider the convex hull Q of A. Then there is a bijection between the H-orbits and the faces of the convex polytope Q. The book is, by the way, also very stimulating for the many beautiful and inspiring pictures it contains of the special polytopes which occur. The idea of considering convex hulls of points with integer coordinates, corresponding to monomials with nonzero coefficients, goes back to Newton, who introduced the so-called Newton diagram in order to produce an algorithm to find local parametrizations of algebraic plane curves (the fractionary power series thus appearing are nowadays called Puiseux series). The Newton diagram of a function is obtained from the points ω corresponding to monomials with nonzero coefficients via taking the convex hull of the union of the sets $\omega + \mathbf{Z}^k$ and has been used with continued success in singularity theory (cf. work of Hironaka, Kouchnirenko, and lately Bierstone and Milman; see e.g. the most recent paper [BM]). There is an obvious link to singularity theory in that considering the cone Y_A over the projective variety X_A is the combinatorial counterpart of viewing A as a subset of $\mathbf{Z}^{k-1} = \{\omega \in \mathbf{Z}^k | \omega_k = 1\}$. In this situation one considers the semigroup S generated by A in $\mathbf{Z}^k$, denotes by $K(S)$ the convex hull of S, and defines the diagram of S, denoted by $K_+(S)$, to be the convex hull of $S - \{0\}$; then $K_-(S) = K(S) - K_+(S)$ is called the subdiagram of S. The above notion is very important since in this way one can analyze the local structure of the singularities of toric varieties and in particular describe the multiplicity of the singular point as the volume of the subdiagram.

The idea of calculating multiplicities is ubiquitous in the theory; for instance, given $k-1$ functions which are general linear combinations of the monomials in A, the number of common roots in $H = (\mathbf{C}^*)^{k-1}$ of these functions equals the volume of the polytope Q in $\mathbf{R}^{k-1}$. This is the theorem of Kouchnirenko, which was applied to calculate Milnor numbers of complete intersections, and holds more generally to calculate Euler Poincaré characteristics of complete intersections of any number $h \leq (k-1)$ of general functions as above. In turn, there is a generalization by Bernstein of the Kouchnirenko theorem, where each function f_i is general among those which are linear combinations of the monomials in a set A_i, and the result is expressed as the mixed volume of the corresponding polytopes Q_i, i.e., as the coefficient of $\lambda_1 ... \lambda_{k-1}$ in the volume of the polytope $\lambda_1 Q_1 + ... \lambda_{k-1} Q_{k-1}$ (this notion goes back to Minkowski).

It is worthwhile noticing that the theory of toric varieties, and of toroidal singularities, was developed in connection with the problem of compactifying the moduli space of Abelian varieties, first by Mumford and coworkers ([KKMS], [AMRT]), then by Oda, Voronoi, etc. Later, it was systematized in an article of Danilov ([D]), and nowadays it is also at the centre of attention as a benchmark for verifying difficult conjectures in particular cases. To give a brief idea of the relation of toric varieties with periods of Abelian varieties, it may suffice to recall that, when we consider complex numbers z which are taken $\mod \mathbf{Z}$, then the complex coordinate $q = \exp(2\pi i z)$ takes values in $\mathbf{C}^*$ and, as z tends to $i\infty$, q tends to 0; in a similar way one compactifies the moduli space of period matrices of Abelian varieties. After this rather lengthy introduction, not much space is left for the description of the main original body of the book, which summarizes the results of about a dozen papers of the authors.

4. Secondary polytopes, A-resultants and A-discriminants

We come now to the crucial part of the book: The notion of toric varieties has made clear what it means to consider special classes of polynomials (sometimes called "sparse" if the number of monomials considered is relatively small). What remains is to consider the associated generalized resultants and discriminants; in the geometric terms previously described, we have to consider the associated Chow form and the dual variety.

In connection with the Chow form R_X of a toric variety X one introduces another convex polytope $Ch(X)$, called the Chow polytope of X. The definition is rather straightforward, since we have a polynomial R_X (the Bertini form), and we simply look at the set S of vectors in $(\mathbf{Z})^N$ which are the exponents of the monomials appearing in R_X with nonzero coefficient: after that we take the convex hull of S. A brief reflection shows that if X is a toric variety, with polytope P, this new polytope depends only on P, and what is very interesting is to describe the polytope $Ch(X)$ in terms of P. It turns out that we get the so-called "secondary polytope" of P, defined through the concept of a coherent triangulation of P. To this purpose, one assumes that A is a finite set whose convex hull is P and considers triangulations of (P, A), i.e., a decomposition of the polytope P as a union of simplices with vertices in A, such that the intersection of two simplices is again a simplex. A triangulation T is said to be coherent if there exists a continuous concave function $g : P \to \mathbf{R}$ such that it is linear on the simplices of T, and the maximal subdomains of linearity are indeed simplices of T. To such a triangulation, one associates a so-called characteristic function of T, $\phi_T : A \to \mathbf{R}$, such that $\phi_T(a)$ is the sum of the volumes of the simplices having the point a as a vertex. Finally, the secondary polytope is defined to be the convex hull (in the space $\mathbf{R}^{\mathbf{A}}$) of such characteristic functions.

One sees that the definition is rather subtle, but indeed the authors show how relevant and meaningful this notion is. In fact, even the simple examples of the coherent triangulations of a segment P in the real line $\mathbf{R}$ (where the finite set A contains more than 2 points) lead to remarkable connections with the representation theory of the Lie algebra of matrices with trace zero, in particular to the solution of a conjecture by Konstant.

Quite interesting are the next examples, which the authors treat in detail: the case of a polygon in the plane, whose sides are labelled $x_1, ...x_n, x_\infty$, leads to all the possible ways of multiplying, in a non-associative composition law, $x_1, ...x_n$ in the given order. This essentially amounts to putting parentheses: e.g., a possible way of multiplying x_1, x_2, x_3, x_4 is $((x_1x_2)\ (x_3x_4))$. In this part the authors instigate the reader to investigate open problems, such as the study of the secondary polytope for a seemingly innocuous example, namely the product of two simplices $\Delta^p \times \Delta^q$ (A is in this case the Cartesian product of the respective sets of vertices). In this example, one is able to understand only the maximal simplices of the polytope in terms of bipartite graphs and some other simplices related to the shuffles of words $A_1...A_pB_1...B_q$ (these are the permutations which leave the relative order of the A_i's, resp. of the B_j's, unchanged). Later in the book, the authors show how shuffles are related to certain "extreme" monomials appearing in the classical Sylvester determinantal expression for the resultant, which were observed by Gordan ("extreme" is now understood as meaning: yielding vertices of the Newton polytope). Finally, concerning more recent results, the hard core of the book is contained in chapters 8-11, dedicated to A-resultants and Chow polytopes, A-discriminants, principal and regular A-determinants and A-discriminants. Again, here A is a subset in $\mathbf{Z}^{k-1}$, and one considers resultants and discriminants for linear combinations of monomials having exponents which are vectors in A. The main tool is a complex of differential forms whose determinant equals the resultant: moreover it is explained that, while the Chow polytope equals the secondary polytope of the convex hull of A, the coherent triangulations correspond to some degenerations of the toric variety X_A under the action of the algebraic torus $H = (\mathbf{C}^*)^{k-1}$.

In chapter 9, one main result is a formula for the degree of the A-discriminant as a sum of volumes of faces of the polytope P spanned by A and a differential geometric characterization of A-discriminantal hypersurfaces. In chapter 10, an allied concept is taken into consideration, i.e. the concept of the principal A-determinant, which is indeed the resultant of f and of the functions x_if_i (f_i is the partial derivative with respect to x_i); for this, the Newton polytope is described as a secondary polytope, and a calculation is provided for the coefficients of the monomials which correspond to vertices of the secondary polytope. Since the principal A-determinant is clearly divisible by the A-discriminant, but contains extra factors, the rather technical chapter 11 is devoted to removing those, finally expressing the A-discriminant as a product, with exponents equal to $+1$ or -1, of principal determinants obtained by taking the intersection of A with certain faces of its convex hull P. The tools used range from the theory of logarithmic de Rham complexes to Whitney stratifications, constructible sheaves, vanishing cycles and holonomic $\mathcal{D}$-modules, making full use of the Riemann-Hilbert correspondence and of ideas of mixed Hodge structures.

5. Brief highlight of other interesting themes

Hypergeometric integrals were the main motivation behind the theory developed in the book, but, as the authors hint, the book grew too much in size for this latter theory to also be developed; quite probably this topic will be taken up in a forthcoming volume. As the authors say, hypergeometric integrals can be viewed as a "quantization" of discriminants (cf. [HLY]). The idea is to study the integral of expressions like $f(x_1, ...x_{k-1})x_1^{a_1}...x_{k-1}^{a_{k-1}}dx_1, ...dx_{k-1}$, for f varying in the space spanned by a certain set of monomials and to express this as a very interesting power series in the coefficients of f (called "hypergeometric" after Gauss' celebrated hypergeometric series, which yields the solution in a very particular case, i.e. the 1-variable case).

On the other hand, plenty of other applications are given, for instance, to the study of domains of convergence of Laurent series, related to the study of the so-called amoeba of a Laurent polynomial f. The amoeba is the image of the zero locus of f in $(\mathbf{C}^*)^k$ under the map $\log : (\mathbf{C}^*)^k \to \mathbf{R}^k$ given by taking the logarithm of the coordinates. The relation to the Newton polytope $N(f)$ here is that the intersection of the amoeba complement with a sphere of big radius M has a limit, given by the codimension 1 skeleton of the triangulation of the sphere determined by the faces of $N(f)$. What pops out here, under the log map, is a particular case of the so-called moment map, defined more generally for a toric variety X_A and with values in the polytope P which is the convex hull of A.

The moment map makes its appearance once more later in connection with real algebraic geometry, especially with Hilbert's problem, which in this context asks for the topological classification of real hypersurfaces corresponding to real polynomials lying in the complement of the A-discriminant. Here, the main theorem is due to Viro and shows that the isotopy type of such a hypersurface has a completely combinatorial description. The idea is that the real part of the toric variety X_A can be glued by copies of the polytope P (the moment map is crucial to construct this homeomorphism), and inside each copy of P one constructs, with the aid of the corresponding coherent triangulation T and an appropriate choice of signs, a very explicit union of cells of the dual subdivision to T. This different proof of Viro's theorem lends itself to analysing the manifold surgeries that one obtains while crossing the discriminantal hypersurface: these have an extremely direct description in terms of the corresponding initial and final coherent triangulations. We find that there is no time here to comment more on other exciting topics, such as Gale transform, Matroids, Shuffles, generating functions associated to discriminant degrees, possible probabilistic interpretations of coefficients of A-discriminants as "entropies", explicit classification (due to Weyman) of the resultants for which Cayley's method of calculating the determinant of a complex reduces to the determinant of a single matrix, and many others.

The final very interesting topic which we would like to briefly introduce is the one of the hyperdeterminants, treated in chapter 14. The role that determinants of square matrices play for the determination of the rank of a matrix is well known. In general, for a k-tensor, the rank is the smallest integer r such that the tensor can be written as the sum of r simple tensors (i.e., those obtained by taking the tensor product of k vectors). More interesting is the so-called border rank, which is the minimal r such that our tensor is in the closure of the tensors of rank r. For algebraic geometers (working over the complex numbers) the condition is simply expressed by saying that the border rank is $\leq r$ if the corresponding point lies in the r-secant variety of the Segre product of the corresponding projective spaces. Here, the hyperdeterminant is defined through the equation of the dual variety to the Segre variety cited above, and many methods, also classical, to compute hyperdeterminants are explained. To tie in with the case of matrices, we only need to remark that in this case the dual variety to the space of rank r matrices (more precisely, rank $\leq r$) is the variety of corank r matrices. It is not clear that an inductive treatment such as the one for matrices can be given also for higher tensors. Finally, the combinatorial characterization of the hyperdeterminant yields relations with the theory of error correcting codes, which the authors mention as an interesting direction of research.

6. Final comments

One of the reasons why the authors are able to cover such vast material in the book is that they tend to develop the main ideas exactly to the right degree of generality that they need. Moreover, the treatment is always done by progressing from the special to the general case, essentially avoiding pedantic verifications. Sometimes the proof is even given through the right picture: the authors have really saved the reader from a lot of unnecessary heavy notation. Moreover, the notation is not only simple, but also everywhere consistent (also in the review I have been trying to use the standard notation of the authors). There are several examples of this tendency to avoid interesting but not useful digressions: for instance, concentrating only on the case of the complex and real numbers, thus e.g. avoiding the pathologies of the duality theorem in positive characteristics (although a general reference is given, [KP]). It is quite pleasant, moreover, to see the basic features of several major theories exposed in a terse way. Another example of this attitude is the treatment of normal singularities: the general standard definition through the algebraic property of the coordinate ring of being integrally closed is recalled, but soon a very precise criterion is given for the case of toric varieties (without proof, but with full reference after a basic example illustrates the meaning of the criterion).

One basic question I would like to answer is whether this book is meant for (graduate) students. My feeling is that first of all it is very nice for students to see so many concrete examples and pictures. The reader here is well motivated by simple but illuminating examples (one such instance being on pages 18-19, the calculation of dual curves in the simplest cases) before being faced with general notions; this is done quite systematically in the book, where examples and pictures always precede the proofs and allow those to be easily stated and easily understood.

There is of course a small price for this; general concepts such as spectra, sheaves, triangulations, and cohomology have to be taken for granted (in fact, the authors do refer to the book [Hart] by Hartshorne), and, although many important concepts of commutative and homological algebra are explained, the authors resist the temptation of launching themselves into interesting generalizations (e.g., Fitting ideals or Eagon-Northcott complexes) which would, however, not be necessary for the needed purposes. The essential ideas of the proofs are always given (an exception occurs only once, for Theorem 1.6 on page 126, which is followed by a "Proof" which actually only proves the first (and easier) assertion).

The above remarks, together with the observation that I could only catch a missing reference ([Whi]), show that actually the book is almost perfectly written, and thus I warmly recommend it not only to scholars but especially to students. The latter do need a text with broader views, which shows that mathematics is not just a sequence of apparently unrelated explosions of new theories, avidly investigated in the beginning and then rapidly forgotten as soon as difficult problems come up, but instead a very huge and intricate building whose edification may sometimes experience difficulties (as the one of the Babel Tower) but eventually progresses steadily. Of course one needs for this purpose expert engineers of the mathematical science who do not get lost with the polishing of a finer detail, but always keep in mind the whole complex architecture, like the writers of the present book.

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