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Introduction to geometric probability, by Daniel A. Klain and Gian-Carlo Rota, Cambridge University Press, 1997, xiv+178 pp., \$59.95, ISBN 0 521 59362 X (hardback), \$19.95, ISBN 0 521 59654 8 (paperback)

It is, of course, tempting to treat Gian-Carlo Rota's last book with a one-sentence dictum in the style of his notorious *Advances* reviews. Here the line might read "These are polished lecture notes in the best sense of the words: if you missed the lectures, then read the notes." But perhaps I could not get away that easily, and it would not really be a very specific review. In any case: these are notes based on Gian-Carlo Rota's 1986 "Lezioni Lincee" lectures about geometric valuations, integral geometry and geometric probability, and they are superbly polished.

The subject of "geometric probability" is not defined in this book. But it appears that a key problem to be treated is the construction of the invariant measures on the sets of points, lines, hyperplanes, etc., in some linear or spherical space. As a counterpart, one tries to understand the collection of all valuations on, for example, the finite unions of convex sets. Among these, the volume and the Euler characteristic stand out. The measures are needed in order to be able to talk about "a random point" or "a random line", and to properly define the probability that a random point will hit a (convex) body, a line will pierce it, and similar unfortunate events.

Thus measures are needed, but why valuations? A key sentence appears in the Notes for the second chapter: "In the sections that follow we shall see that most of the interesting functionals of geometric probability satisfy the valuation property in some respect." This comes after one has seen (in the opening lecture) a stunning solution to Buffon's needle problem, "What is the probability that a short needle dropped on ruled paper comes to lie in a position that crosses a line?"—a solution that derives the (surprising) answer without evaluating any integrals, but by setting up a valuation. In the following lectures, various striking results are derived and values are computed — using very little machinery besides the "power of valuations." Again and again the pattern is the following: first construct valuations using Groemer's integral theorem, then characterize the invariant valuations by comparing them with suitable linear combinations of the examples that were already constructed, and then use this characterization as a structure theorem for any invariant valuations that occur "in nature". This is then applied in various striking ways.

The exposition is marvellous: clear and precise (and brief — so it requires concentrated reading). The powerful theory of valuations, intrinsic volumes and invariant measures built by Hadwiger, Groemer, McMullen and others is an impressive development. The path led by Klain and Rota proceeds from the simple and special to more powerful and general results, like a guided ten-day tour into a wild mountain range. Each daily hike leads to another peak and opens the view to others. A main result reached is Hadwiger's 1957 characterization of the invariant measures on $\mathbb{R}^n$, here presented with a new proof due to Daniel Klain; applications include the

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(Dehn's) solution of Hilbert's third problem on non-equidecomposable polyhedra and Lutwak's containment theorem via Helly's theorem.

The beautiful exposition would make this volume worthwhile even if Klain and Rota hadn't "something new" to say. I can see two driving forces behind their exposition. One is the analogy between the discrete case, where $P(S)$ is the distributive lattice of subsets of an n-element set S, and the continuous case, where $Mod(n)$ is the modular lattice of all linear subspaces of $\mathbb{R}^n$. Thus, for example, we meet a discrete version of Helly's theorem first, and only later the better-known version for convex sets.

The second point is to push their discovery of "the right" normalization for the invariant measures on the Grassmannian

$$\nu_k^n(\mathrm{Gr}(n,k)) \ := \ \binom{n}{k}\frac{\omega_n}{\omega_k\omega_{n-k}}$$

— a version whose power is demonstrated in Cauchy's surface area formula (where the normalization constant vanishes), in a natural continuous version of Sperner's Lemma (about the maximal measure for an antichain of subspaces in $\mathbb{R}^n$), etc.

How much one can learn in the course of these lectures becomes apparent in Section 9.6, where a vast generalization of Buffon's needle problem is established with great ease. On the other hand, a sequence of eleven lectures doesn't of course "teach you the whole subject": rather, it should (and does) motivate you to read more. For this the Notes to each chapter and the extensive bibliography give many good pointers, among them classics such as Hadwiger's [1] treatise and the current survey by McMullen [2] as two definitive sources.

The cover text advertises this little red book as "a stimulating and fruitful tale." It is.

References

[1] H. Hadwiger: Vorlesungen über Inhalt, Oberfläche und Isoperimetrie, Springer-Verlag, Berlin, 1957. MR **21:**1561

[2] P. McMullen: Valuations and dissections, Chapter 9.3 in: "Handbook of Convex Geometry" (P.M. Gruber, J.M. Wills, eds.), Vol. B, North-Holland, Amsterdam, 1993, pp. 933-988. MR **95f:**52018

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