BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 3, Pages 343–349
S 0273-0979(00)00864-8
Article electronically published on April 3, 2000

Representation theory and complex geometry, by Neil Chriss and Victor Ginzburg, Birkhäuser, Boston, 1997, x + 495 pp., $64.50, ISBN 0-8176-3792-3

1. Geometric representation theory

The book belongs to the "geometric representation theory", and this geometric view will make it attractive and accessible to an audience beyond the representation theorists.

Representation theory. Let us try to study a given object $\mathfrak{X}$ through vector spaces V associated naturally to it. The symmetries $\mathcal{S}$ of $\mathfrak{X}$ can form a group or more generally some kind of an algebra usually related to a group (Lie algebra, quantum group,...). The Vector space V inherits the symmetries of $\mathfrak{X}$, and they act on V by linear operators. This is what one means by "V is a representation of the structure $\mathcal{S}$". A representation theorist produces the list of all irreducible representations of $\mathcal{S}$ (those that do not admit a non-trivial $\mathcal{S}$-invariant subspace, i.e., the simplest ones), studies irreducible representations in detail and reconstructs interesting representations V by gluing together the irreducible ones.

Two sources of interest in representation theory are particle physics and the Langlands conjectures in number theory. While the traditional representation theorist is primarily an algebraist, most of the successes in the Langlands program use some geometric methods.

Geometry. The idea is that in order to study a given object A, one looks for a geometric object that encodes information on A. This allows the use of a variety of geometric techniques such as topology of algebraic varieties (homology, K-theory, Hodge theory, Grothendieck groups of sheaves, mixed sheaves) or modules over the sheaf of rings of differential operators ("D-modules"). As I cannot give much of a unifying view on applications of this strategy, I will list a few examples that are close to the spirit of the book. However, we first have to recall

What are *D*-modules and perverse sheaves? The notion of D-modules gives an algebraic framework for linear differential equations. On the other hand, the notion of perverse sheaves arises as the "correct" point of view on solutions of such equations [BEGKHM].

When one considers a system of linear differential equations on a complex manifold X, $d_i(f)=0,\ i\in I$, one can associate to it a module M over the ring D_X of linear differential operators on X. One has $M = D_X/\sum_{i\in I} d_i{\cdot}D_X$, and the analytic solutions of the system can be identified with $Sol(M) = \mathrm{Hom}_{D_X}[M,\mathcal{O}_X^{an}(X)]$. To capture the local solutions we pass to the sheaf $\mathcal{H}om_{\mathcal{D}_X}(M,\mathcal{O}_X^{an})$. The first product of the algebraic reformulation is an improved notion of solutions. We replace the functor $\mathcal{H}om$ with its derived version to get a complex $\mathcal{S}ol(M)\overset{\text{def}}{=} \mathrm{R}\mathcal{H}om_{D_X}(M,\mathcal{O}_X^{an})$.

If our D-module M is small enough ("holonomic"), then $\mathcal{H}om_{\mathcal{D}_X}(M,\mathcal{O}_X^{an})$ is a constructible sheaf; i.e., there is a stratification of X (a decomposition into smooth algebraic subvarieties $X = \sqcup X_j$), such that the sheaf is locally constant on each

2000 *Mathematics Subject Classification.* Primary 22-XX; Secondary 14-XX.

stratum X_j. Holonomic D-modules contain a subclass of "modules with regular singularities" which are completely determined by their analytic solutions – an appropriate category $D_{RS}(X)$ of complexes of D-modules with regular singularities is equivalent via the solution functor $\mathcal{S}ol$ to an appropriate category $D_c(X)$ of complexes of constructible sheaves on X (Riemann–Hilbert correspondence). The so-called perverse sheaves are just the solution complexes of $\mathcal{D}_X$-modules with regular singularities. They form an abelian subcategory $\mathcal{P}(X)$ of $D_c(X)$. Perverse sheaves also have a purely topological characterization. As a consequence, for any commutative ring $\boldsymbol{k}$ one has a category $\mathcal{P}(X, \boldsymbol{k})$ of perverse sheaves with coefficients in modules over $\boldsymbol{k}$ (our $\mathcal{P}(X)$ is now $\mathcal{P}(X, \mathbb{C})$). For the same reason one has perverse sheaves on any algebraic variety over a field (in etale topology and with l-adic coefficients) [BBD].

2. Study of functions

Let us examine the case when the object we want to study by geometric methods is just a function.

D-modules. To study a function f on a complex manifold X via the linear differential equations it satisfies is the same as considering the module M_f over the ring of differential operators D_X on X generated by f. The module M_f consists of all functions obtained by applying differential operators d to f, and the kernel of the surjective map $D_X \to M_f$ consists of all linear differential operators $d \in D_X$ that kill f. An interesting function f will often satisfy enough linear differential equations to make the D-module $M_f = D_X \cdot f$ small enough so that it can be efficiently analyzed.

Constructible sheaves. It turns out that on an algebraic variety X defined over an algebraically closed field $\mathbb{F}$ of positive characteristic there is a similar and even closer relation between sheaves and functions ("Grothendieck's function–sheaf dictionary"). Suppose that X is defined over a finite field $\mathbb{F}_q$ with q elements. Let $\mathcal{M}$ be a constructible sheaf on X (in etale topology), which is also defined over $\mathbb{F}_q$ in the sense that one has an isomorphism $\phi : \mathcal{M} \xrightarrow{\cong} (Fr_{X,\mathbb{F}_q})^* \mathcal{M}$ with the pull-back of $\mathcal{M}$ under the Frobenius map $Fr_{X,\mathbb{F}_q} : X \to X$. Then $\mathcal{M}$ defines a function f on the finite set $X(\mathbb{F}_q)$ of points of X with coordinates in $\mathbb{F}_q$, where the value at a point x is the trace of ϕ on the stalk of $\mathcal{M}$ at x. Moreover, we get in the same way a family of functions f_n on sets $X(\mathbb{F}_{q^n})$ for $n = 1, 2, \ldots$.

One can think of $D_c(X)$ as a "categorification" of the vector space of functions on X. Any map $X \xrightarrow{\pi} Y$ defines the linear operators of "pull-back" and "integration over fibers", and these lift to functors $D_c(Y) \underset{\pi_!}{\overset{\pi^*}{\rightleftarrows}} D_c(X)$. So the two settings behave the same, but $D_c(X)$ is a much richer object provided with new tools. For instance, there is the "Verdier duality" functor (contravariant) $\mathbb{D}_X : D_c(X) \to D_c(X)$ which when X is a point reduces to the duality on vector spaces.

Character sheaves. Lusztig's theory of character sheaves [Lu1] deals with the calculation of character functions $\Theta_V(g) \overset{\text{def}}{=} \mathrm{Tr}_V(g)$ of irreducible representations V of "finite groups of Lie type", i.e., groups $G(\mathbb{F}_q)$ where G is a split reductive group defined over a finite field $\mathbb{F}_q$. One starts by constructing a class of irreducible perverse sheaves on G and proves that the associated functions (which are algorithmically computable) give another basis of the vector space spanned by the irreducible characters. The end-product at the moment is the conjectural formula for the transition matrix, known to be correct in many cases.

Automorphic sheaves. Similarly, there is a program due to Drinfeld for constructing "automorphic sheaves". In positive characteristic these should conjecturally give all unramified automorphic forms predicted by the Langlands conjectures, and this would finish the proof of these conjectures in positive characteristic. The constructions of automorphic sheaves over complex numbers by Beilinson–Drinfeld are based on their geometric formulation of a large part of the formalism of the conformal field theory ("chiral algebras") [BD].

3. Geometric localization

If a Lie algebra $\mathfrak{g}$ acts on a manifold X (i.e., one has a map of Lie algebras from $\mathfrak{g}$ to the vector fields on X), global sections of any $\mathcal{D}_X$-module $\mathcal{M}$ carry an action of $\mathfrak{g}$. In the opposite direction, any $\mathfrak{g}$-module M produces a D-module $\nabla(M) = \mathcal{D}_X \underset{\mathfrak{g}}{\otimes} M$, so M is encoded as a sheaf and can be studied locally on X.

Flag variety. In the case when $\mathfrak{g}$ and X are the Lie algebra and the flag variety of a complex reductive group G, the functors of localization and global sections are inverse equivalences of categories of $\mathcal{D}_X$-modules and of $\mathfrak{g}$-modules satisfying certain conditions (which can be removed by considering sheaves or rings that are locally isomorphic to $\mathcal{D}_X$) [BB]. This is a vast generalization of the Borel–Weil realization of irreducible finite dimensional representations as sections of line bundles on flag varieties.

To see the strength of the method, consider the (usually infinite-dimensional) representations of real reductive groups. A real form $G_{\mathbb{R}}$ of G can be algebraically encoded as an algebraic subgroup $K \subseteq G$ with finitely many orbits on X, so that the $\mathcal{D}_X$-modules which are K-equivariant correspond to representations of $G_{\mathbb{R}}$. In this way, one obtains an algebro-geometric picture for the representation theory of $G_{\mathbb{R}}$ (X stratified by the K-orbits). Next, the Riemann–Hilbert correspondence yields a topological description of irreducible $\mathcal{D}_X$-modules in terms of the intersection homology of orbit closures, which in turn can be calculated using the Decomposition Theorem, a deep result of Hodge theory (or the theory of mixed sheaves). This description of intersection homology is precisely the character formula for irreducible $G_{\mathbb{R}}$-representations that was conjectured by Kazdhan–Lusztig–Vogan and proved in this way by Beilinson–Bernstein [BB] and Brylinski–Kashiwara.

Moduli of curves. A more sophisticated example is the action of the Virasoro Lie algebra on the moduli of curves, actually on the determinant line bundle over the moduli of curves with additional data (local parameters at marked points). The corresponding localization yields a construction of field theories [BFM].

4. Topological localization

A much more mysterious localization of finite dimensional algebraic representations of a complex reductive group appears in the following way [Gi] (this is an idea of Drinfeld that was completed by Ginzburg using a key result of Lusztig). The loop Grassmannian of a complex reductive group G is the homogeneous space $\mathcal{G} = G(\mathbb{C}[z, z^{-1}])/G(\mathbb{C}[z])$ of the corresponding loop group $G(\mathbb{C}[z, z^{-1}])$. Consider the category $\mathcal{P}_{G(\mathcal{O})}(\mathcal{G},\mathbb{C})$ of perverse sheaves with complex coefficients on $\mathcal{G}$ that are equivariant with respect to the disc group $G(\mathbb{C}[z])$. It turns out that it is equivalent to the category of representations of another reductive complex group $\check{G}$ which does not act on $\mathcal{G}$. Moreover, for any commutative ring $\boldsymbol{k}$ the category $\mathcal{P}_{G(\mathcal{O})}(\mathcal{G},\boldsymbol{k})$ of perverse sheaves with $\boldsymbol{k}$-coefficients is equivalent to the category of algebraic representations of the split $\boldsymbol{k}$-form of $\check{G}$ [MV]. The group $\check{G}$ is called the Langlands dual of G. It has a simple combinatorial description in terms of G, but the relation to G (the basic question in the Langlands program) is deep. For instance, if $G = GL(V)$, the above construction (which provides the only known direct construction of $\check{G}$ from G) describes $\check{G}$ canonically as $GL(\check{V})$ where $\check{V} = H^*[\mathbb{P}(V),\mathbb{C}]$ is the cohomology of the projectivization of V.

The perverse sheaves in our category are geometrizations of Hecke operators, so the above picture appears already in the definition of automorphic sheaves. The use of $\boldsymbol{k}$-coefficients gives a geometric approach to representations of reductive groups defined over fields of positive characteristic ("modular representations") and even over integers. (Since for $H = \check{G}$, $\check{H}$ is isomorphic to G, the method applies to representations of arbitrary reductive groups.)

5. Convolution algebras

An interesting algebra often has a set-theoretic model – it can be realized by convolution of correspondences. Such a model may provide an insight into its representations.

Convolution of correspondences. A finite set X defines a vector space $\mathbb{C}[X]$ of complex valued functions on X. A map of finite sets $\pi: X \to Y$ defines obvious linear maps $\mathbb{C}[X] \xrightarrow{\pi_!} \mathbb{C}[Y] \xrightarrow{\pi^*} \mathbb{C}[X]$, and therefore any correspondence $\mathcal{C} = (X \xleftarrow{f} C \xrightarrow{g} Y)$ defines an operator $[\mathcal{C}] = g_! f^*: \mathbb{C}[X] \to \mathbb{C}[Y]$. The composition of operators $[\mathcal{C}]$ corresponds to the convolution of correspondences $\mathcal{C}$:

$$(X \xleftarrow{\alpha} A \xrightarrow{\beta} Y) * (Y \xleftarrow{\gamma} B \xrightarrow{\delta} Z) \stackrel{\text{def}}{=} (X \xleftarrow{f} C \xrightarrow{g} Z)$$

with $C = \{(a,b) \in A \times B,\ \beta(a) = \gamma(b)\}$.

A simple example. Let $Gr(X)$ be the set of all subsets of a set X with n elements. Let $\mathcal{E}$ be the correspondence on $Gr(X)$ given by pairs of subsets (S,T) with $S \supseteq T$ and $|S - T| = 1$, and let $\mathcal{F}$ be the opposite correspondence. It turns out that the linear operators $e = [\mathcal{E}]$ and $f = [\mathcal{F}]$ generate an action of the Lie algebra $\mathfrak{sl}(2)$ on $\mathbb{C}[Gr(X)]$. The subspace of functions that depend only on the size of $S \subseteq X$ forms an $(n+1)$-dimensional irreducible subrepresentation, and for $n = 0, 1, \ldots$ we get all irreducible finite dimensional representations of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ or the group $SL(2,\mathbb{C})$. The action of $\mathfrak{sl}(2)$ on $\mathbb{C}[Gr(X)]$ commutes with the obvious action of the group S_n of permutations of X (these are just the obvious actions on $(\mathbb{C}^2)^{\otimes n}$).

"Change of coefficients". The same set-theoretic setting may be used to produce convolution algebras related to different representation theories. In the above example let us replace X with a vector space V over a field $\mathbb{F}$ having a basis X. Instead of subsets of X consider the Grassmannian $\mathcal{G}r(V)$ of all linear subspaces of V. If $\mathbb{F}$ is chosen to be a finite field with q-elements, the analogous correspondences generate an action of the quantum group $SL(2)_q$ on $\mathbb{C}[\mathcal{G}r(V)]$ that commutes with the obvious action of a finite group $GL_n(\mathbb{F})$. If $\mathbb{F}$ is the field of complex numbers, the sets in question are no longer finite. However, we can ensure that the convolution operators are still defined by replacing the space of functions $\mathbb{C}[\mathcal{G}r(V)]$ with $\mathcal{A}[\mathcal{G}r(V)]$ where $\mathcal{A}$ can be (equivariant) homology, K-theory, or some similar object that one can attach to an algebraic variety.

Convolution algebras from the Chriss–Ginzburg book. Instead of describing generators of an algebra we can often construct the whole algebra. To any map $\mu: \widetilde{\mathcal{N}} \to \mathcal{N}$ one can attach a correspondence $\mathcal{C} = \{(x,y) \in \widetilde{\mathcal{N}}^2,\ \mu(x) = \mu(y)\}$. If μ is proper and $\widetilde{\mathcal{N}}$ smooth, for various choices of $\mathcal{A}$ (as above) the canonical projection $\mathcal{C} * \mathcal{C} \to \mathcal{C}$ makes $\mathcal{A}[\mathcal{C}]$ into an associative algebra. If $\mathcal{A}$ is the Borel–Moore homology, beautiful ideas of Ginzburg provide a classification of irreducible modules of $\mathcal{A}[\mathcal{C}]$ (roughly parameterized by the types of fibers of μ).

The main choice of μ in the book is the moment map $T^*X \to \mathfrak{g}^*$ from the cotangent bundle over the flag variety X of a reductive group G to the dual of its Lie algebra $\mathfrak{g}$. In this case the Borel–Moore homology $\mathcal{H}_*(\mathcal{C})$ is the group algebra of the Weyl group W of G, and one recovers Springer's construction of irreducible representations of W. (Say, for $G = GL_n$ the Weyl group is the permutation group S_n.)

Moreover, the equivariant K-group $K^{G \times \mathbb{C}^*}(\mathcal{C})$ is affine Hecke algebra $H^{\text{aff}}(\check{G})$ attached to the Langlands dual group $\check{G}$. The resulting classification of irreducible representations of $H^{\text{aff}}(\check{G})$ is one of the principal successes of the Langlands program and the high point of this book. The approach here follows the original proof by Kazdhan and Lusztig with beautiful improvements.

The third representation-theoretic topic of the book is the description of the irreducible finite dimensional representations of $\mathfrak{sl}(n)$. One uses the Borel–Moore homology and (for a single n) a combination of moment maps for all $\mathfrak{sl}(m)$'s – this is a geometric version of the Schur-Weyl description of irreducibles for $\mathfrak{sl}(n)$ in terms of all symmetric groups S_m. If one passes in this setting to the equivariant K-theory, $\mathfrak{sl}(n)$ is replaced by its quantum affine version [GV]. An extension of the construction to all semi-simple Lie algebras (and much more) is found in Nakajima's treatment of quivers [Na].

6. Grothendieck groups

Our basic strategy is to realize an interesting abelian group A as the Grothendieck group $K(\mathcal{A})$ of an abelian category $\mathcal{A}$; then the properties of the category $\mathcal{A}$ may tell us something new about A. Our example will show that any irreducible finite dimensional representation of a semi-simple complex Lie algebra $\mathfrak{g}$ has a (non-trivial and very useful) canonical basis – completely unexpected news after 50 years of mastery of the subject! (For simplicity let $\mathfrak{g}$ be "simply laced", for instance $\mathfrak{g} = \mathfrak{sl}(n)$.)

6.1. **Quivers.** Consider a quiver (directed graph) Γ with vertices I, say $1 \to 2 \to \cdots \to n$. A representation π of Γ over a field $\mathbb{F}$ is a way of attaching to each edge $i \xrightarrow{e}$ a linear map $\mathbb{F}^{d_i} \xrightarrow{\pi(e)} \mathbb{F}^{d_j}$. Representations of dimension $d = (d_i)_{i \in I}$ form an algebraic variety Rep_d with an action of $GL(d) = \prod_{i \in I} GL(d_i)$. If $\mathbb{F}$ is a finite field $\mathbb{F}_q$, consider the vector spaces $\mathbb{C}_{GL(d)}[Rep_d]$ of $GL(d)$-invariant functions. Ringel observed that the sum over all d's is an algebra with the multiplication given by an obvious "equivariant convolution" of functions [Ri]. If the corresponding non-directed graph is a Dynkin diagram of a simple Lie algebra $\mathfrak{g}$ $(1 \to \cdots \to n$

corresponds to $\mathfrak{g} = \mathfrak{sl}(n+1)$), this algebra is the "positive part" of the quantum group $\mathfrak{g}_q$, i.e., the quantization of a maximal nilpotent Lie subalgebra $\mathfrak{n}\subseteq\mathfrak{g}$ [Ri].

6.2. **Canonical bases** [Lu2]**.** Now let $\mathbb{F} = \mathbb{C}$ and, following Lusztig, "change the coefficients" by replacing the invariant functions with the Grothendieck group of equivariant perverse sheaves $K(\mathcal{P}_{GL(d)}[Rep_d])$. The sum over d's is the enveloping algebra $U(\mathfrak{n})$ of $\mathfrak{n}$ (mixed perverse sheaves give the same story for quantum groups). However, our Grothendieck group has a canonical basis given by the irreducible perverse sheaves. So $U(\mathfrak{n})$ has a canonical basis, and it soon follows that so does any irreducible finite dimensional representation of the Lie algebra $\mathfrak{g}$. One of the consequences is that inside the real points $G(\mathbb{R})$ there is an interesting positive part $G(\mathbb{R})_{\geq 0}$, a semigroup consisting of all elements g which act in canonical bases of irreducible representations by matrices with entries ≥ 0 ("G is defined over the semiring $\mathbb{Z}_+$ of non-negative integers").

6.3. **Categorification.** We see that a realization of an object $\mathcal{S}$ as a Grothendieck group of a category $\mathcal{C}$ may be useful in studying $\mathcal{S}$. However, the properties of $\mathcal{S}$ are likely to have more subtle analogues in $\mathcal{C}$, and once we understand those, we have a "categorification" of the structure $\mathcal{S}$ which may be an even more interesting object. From this point of view if an abelian group A has an interesting canonical basis, one should view it as evidence that it may have a categorification. The origins of this point of view are categorifications of constructions of topological quantum field theories in order to get a higher dimensional theory [CF]. One of the successes is Khovanov's categorification of the Jones polynomial, a celebrated invariant of knots [Kh]. Khovanov's "knot homology" attaches to a knot a bigraded abelian group $H = \oplus_{i,j} H^{i.j}$ so that the graded Euler characteristic $\sum (-1)^i q^j \operatorname{rank}(H^{i.j})$ is the Jones polynomial. Obviously, H contains more information, for instance the torsion part.

7. The book

The book gives a uniform geometric approach (via convolution algebras) to the classification of irreducible finite dimensional representations of (1) Weyl groups, (2) Lie algebras $\mathfrak{sl}(n,\mathbb{C})$ and (3) affine Hecke algebras. To those who are familiar with the classical algebraic solutions to (1) and (2), and do not require (3) at the moment, the book should still open a new world, deep and beautiful.

The book is largely self contained with a few exceptions such as the Riemann–Roch theorem. There is a nice introduction to symplectic geometry and a charming exposition of equivariant K-theory. Both are enlivened by examples related to groups. In this book reductive groups are seen through their reflections in the geometry of flag varieties and nilpotent cones, an unreasonably lively and deep topic. A sketch of the formalism of perverse sheaves (and their use) is delegated to the last chapter so that one has a good idea what they will be good for, before going through the formalism. Strangely, there is no index nor a list of notations.

An attractive feature of the book is the attempt to convey some informal "wisdom", rather than only the precise definitions. As a number of results are due to the authors, one finds some of the original excitement. This is the only available introduction to geometric representation theory (see also [Lu3] for an overview of a large part of the subject). The best recommendation is the fact that it has already proved successful in introducing a new generation to the subject.

References

[BB] A. Beilinson, J. Bernstein, A proof of Jantzen conjectures, *Adv. in Soviet Math.* **16**, part 1 (1993), 1-50. MR **95a:**22022

[BBD] A. Beilinson, J. Bernstein, P. Deligne, Faisceaux Pervers, *Astérisque* **100** (1982). MR **86g:**32015

[BD] A. A. Beilinson, V. G. Drinfeld, Quantization of Hitchin's fibration and Langlands' program. Math. Phys. Stud. **19**, Kluwer Acad. Publ. (1996), 3-7. Chiral algebras. MR **97m:**14008

[BFM] A. Beilinson, B. Feigin, B. Mazur, Introduction to algebraic field theories on curves, preprint.

[BEGKHM] A. Borel, F. Ehlers, P. Grivel, B. Kaup, A. Haefliger, B. Malgrange, Algebraic D-modules, Persp. in Math. **2**, Acad. Press, Inc. (1987). MR **89g:**32014

[CF] L. Crane, I. B. Frenkel. Four dimensional topological quantum field theory, Hopf categories, and the canonical bases. *J. Math. Phys.* **35** (1994). MR **96d:**57019

[Gi] V. Ginzburg, Perverse sheaves on a loop group and Langlands duality, preprint alg-geom/9511007.

[GV] V. Ginzburg, E. Vasserot, Langlands reciprocity for affine quantum groups of type A_n, *Internat. Math. Res. Notices* (1993), 67-85. MR **94j:**17011

[Kh] M. Khovanov, A categorification of the Jones polynomial, Preprint math.QA/9908171 (1999).

[Lu1] G. Lusztig, Character sheaves I, II, III, IV, V, *Adv. Math.* **56** (1985), 193-237, **57** (1985), 226-265 and 266-315, **59** (1986), 1-63, **61** (1986), 103-155. MR **87b:**20055, MR **87m:**20118a, MR **87m:**20118b, MR **87m:**20118c

[Lu2] G. Lusztig, Introduction to quantum groups, Boston, Birkhäuser, 1993. MR **94m:**17016

[Lu3] G. Lusztig, Intersection cohomology methods in representation theory, in Proceedings of the International Congress of Mathematicians, Kyoto, Springer-Verlag (1991). MR **93e:**20059

[MV] I. Mirković, K. Vilonen, Perverse sheaves on loop Grassmannians and Langlands duality, Preprint alg-geom/9703010.

[Na] H. Nakajima, Quiver varieties and Kac-Moody algebras, *Duke Math. J.* **91** (1998), 515-560. MR **99b:**17033

[Ri] C. M. Ringel, Hall algebras and quantum groups, *Invent. Math.* **101** (1990), 583-592. MR **91i:**16024

Ivan Mirković
University of Massachusetts at Amherst
E-mail address: mirkovic@math.umass.edu