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Partial differential equations, by L. C. Evans, Grad. Stud. Math., vol. 19, Amer. Math. Soc., Providence, RI, 1998, xvii + 662 pp., $75.00, ISBN 0-8218-0772-2

The subject of Partial Differential Equations fascinates me because the problems are intrinsically interesting and have applications inside and outside mathematics, and because the variety and originality of the tools and ideas involved never cease to surprise me.

The basic problem is as simple as can be. A function whose partial derivatives are constrained to satisfy one or more equations

$$F(x, u, Du, D^2u, \ldots, D^m u) = 0 \tag{1}$$

has a structure which is limited by the nature of the equations. The goal is to get insight into the limitations from the equations.

One begins with some encouraging examples. Functions satisfying

$$u_{x_1}(x_1, \ldots, x_d) = 0 \tag{2}$$

are those which are independent of x_1. The solutions of

$$u_{x_1 x_2} = 0 \tag{3}$$

are sums of two functions each independent of one of the first two variables. The functions $u(t, x)$ with scalar x satisfying

$$u_t + c u_x = 0\,, \qquad c \in \mathbb{R} \tag{4}$$

are constant on the lines moving with speed c.

This heady feeling evaporates quickly. The solutions of Laplace's equation

$$\Delta u \;:=\; \sum u_{x_j x_j} = 0$$

or the Cauchy Riemann equation

$$u_t = i u_x \tag{5}$$

are automatically real analytic. This is in no way obvious when one thinks in terms of the simple relation among the partial derivatives.

There are a few general approaches to analyzing differential equations. The main ones can be described as elementary calculus, the multiplier method, and transformation to a simpler problem. These are not disjoint categories.

Examples of the first class are given above. The most impressive first-year textbook application is Hopf's maximum principle for second order elliptic equations. The analysis starts with the remark that if a real-valued function u has a local maximum, then the pure second derivatives $u_{x_j x_j}$ must be nonpositive, so in particular the Laplacian must be nonpositive. An end result of the analysis is that a harmonic function on a bounded domain must achieve its extrema at the boundary, and at maxima the outward normal derivative is strictly positive.

The multiplier method is a strategy for extracting information by multiplying (1) by a suitable function and then integrating. Each time that one does this, one uses the equation, so that the more multipliers that one uses, the more information

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one may extract. The art is in the choice of multipliers which often depend on the solution itself. Here are two applications of the multiplier method to the heat equation

$$u_t = \sum_j u_{x_j x_j} \,. \tag{6}$$

Suppose that u tends to zero sufficiently rapidly at $x \to \infty$. Multiplying by u and integrating over $[0,t] \times \mathbb{R}^d$ yield

$$0 = \int_0^t \int_{\mathbb{R}^d} u \Big(u_t - \sum_j u_{x_j x_j} \Big)\, dt\, dx \,.$$

Writing the integrand in divergence form

$$\partial_t \frac{u^2}{2} - \sum_j \partial_j (u\, u_{x_j}) + \sum_j (u_{x_j})^2$$

and applying the fundamental theorem of calculus yield

$$\frac{1}{2} \int_{\mathbb{R}^d} u(t,x)^2\, dx + \int_0^t \int_{\mathbb{R}^d} \sum_j (u_{x_j})^2 \; dx\, dt \;=\; \frac{1}{2} \int_{\mathbb{R}^d} u(0,x)^2\, dx \,. \tag{7}$$

This identity shows that the $L^2(\mathbb{R}^d)$ norm of $u(t)$ is a decreasing function of time and the $\nabla_x u$ is square integrable over $\mathbb{R}_+ \times \mathbb{R}^d$. Such estimates are the keys to analyzing partial differential equations. They can be used to prove existence, uniqueness, and continuous dependence results as well as to provide other qualitative information. A puzzle here is how one guessed the multiplier u, especially in view of the fact that the quantity $\int u^2 dx$ does not have a natural physical interpretation. There is no simple answer to the question, but the heat equation does not live in isolation, and reasoning by induction and analogy are both powerful aids.

If one multiplies the heat equation by $e^{-ix.\xi}$ and integrates dx, one obtains an identity for the Fourier Transform $\hat{u}(t,\xi)$ of $u(t,x)$,

$$(2\pi)^{d/2}\, \hat{u}_t(t,\xi) := \int_{\mathbb{R}^d} u_t(t,x)\, e^{-ix.\xi}\; dx = \int_{\mathbb{R}^d} e^{-ix.\xi} \sum u_{x_j x_j}\; dx \,.$$

Integrating twice by parts in the last integral yields the simple ordinary differential equation

$$\partial_t \hat{u}(t,\xi) = -|\xi|^2\, \hat{u}(t,\xi) \,, \tag{8}$$

and thereby the explicit solution formula

$$\hat{u}(t,\xi) = e^{-t|\xi|^2}\, \hat{u}(0,\xi) \,. \tag{9}$$

Note that the argument leading to (7) involved only one multiplier. The identity (7) contains much less information than the original equation. The argument leading to (8-9) used an infinite family of multipliers parametrized by $\xi \in \mathbb{R}^d$. Using all those multipliers, one has not lost any information at all.

The last example is also an example of simplification. Changing the dependent variable to its Fourier Transform in x transforms the heat equation to ordinary differential equations in time. In a similar vein a simple linear change of independent variable transforms (4) to (2). More subtle changes of dependent and independent variable—for example hodograph, Hopf-Cole, and inverse spectral transformations—linearize nonlinear problems, thereby gaining access to some strongly nonlinear phenomena.

Other examples of simplification by transformation involve symmetries of one sort or another. Seeking spherically symmetric solutions of Laplace's equation or radial solutions of the heat equation invariant under the scaling law $u \to u(\lambda^2 t, \lambda x)$ leads to ordinary differential equations whose explicit solution yields the fundamental solutions of the partial differential equations. Searching for special solutions, often guided by invariance principles, is a reasonable first attack on a differential equation. Plane wave solutions of equations with constant coefficients, simple waves for conservation laws, and Barenblatt's solution of the porous medium equation are other examples of this sort discussed in Evans' book.

Another strategy for simplification is asymptotic analysis. Introducing a family of equations or data parametrized by a small parameter sometimes leads to simplified equations describing the limit as the parameter tends to zero. The cases of geometric optics and homogenization are discussed in the text.

Evans' tightly written book has excellent balance between linear and nonlinear equations. It has careful proofs which are sometimes real improvements on those available elsewhere. It includes, as it must, the classic operators and their fundamental solutions, a solid coverage of the basic methods involving energy integrals, Fourier Transform, and maximum principles. In addition it includes topics reflecting the author's interests, notably viscosity solutions of Hamilton-Jacobi equations, entropy solutions of conservation laws, and both weak convergence and topological methods in the calculus of variations. It shares with Courant's classic the properties that one can dive in at almost any topic, and that topics are often treated once and then treated later in more depth. The fact that the book can be read locally permits its use as a text even though there is way more than one could imagine covering in any one-year course. The same quality makes the book a valuable reference. It is one of the most consulted volumes on my shelves. It introduces you to topics and refers you to specialized works for the last word.

There are some choices that the author has made which are worth mentioning. The most notable is that the language of the Theory of Distributions is not used. Sobolev spaces and their notion of weak derivative are employed systematically. Where this becomes awkward is for the fundamental solutions where the delta function on the right is presented as a measure and the differential equation with delta function on the right hand side is described as "formal". The fact of being a fundamental solution is expressed by showing that convolution solves the inhomogeneous equation. In the same vein, solution by Fourier Transform cannot be pushed to its natural domain, for example computing directly such fundamental solutions. The reason for making such a choice is to reduce the number of things a student must learn. My taste is that Distribution Theory has become so ubiquitous in analysis and geometry that one of the good things that a course in Partial Differential Equations does is to familiarize students with this subject.

The book begins with four chapters which are meant to introduce students to basic examples and computations. This comprises 235 pages at a breathtaking pace. I read straight through without omission, and that is clearly not possible for students. The material is simply too dense. In classroom practice one would select a smallish subset of these topics, at a slower pace with more discussion. Too often in these pages an important formula is derived and the discussion of its implications is postponed. There are many loose threads at this point which are tidied up later. An example close to my heart is that there is a loss of derivatives from focusing in the higher dimensional wave equation. This follows from the solution formulas and is not mentioned when those formulas are derived. On the other hand, the Hopf-Lax formulas and the reasons for selecting the entropy and viscosity solutions of conservation laws and Hamilton-Jacobi equations become much clearer some 500 pages later.

I would have liked it if Evans had pointed out that the reality hypothesis in Hamilton-Jacobi theory is exactly a hyperbolicity assumption analogous to that imposed in the theory of first order systems. On the other hand, the text is full of remarks of this sort, and a reader should pay careful attention to these remarks and discussions, as they are very informative and well done.

There are good problems, but I would like more of them.

Methods useful for strongly nonlinear problems, for example variational methods and methods based on convexity and monotonicity, receive a thorough treatment as does the theory of Hamilton-Jacobi equations. For the latter, the basic existence theory is derived from optimal control at the end of the book. This is a particularly pleasing closure to a theory introduced earlier.

Evans' text is appropriate for graduate students with a good background in real analysis. For a long time that niche was dominated by books which were very strongly influenced by Courant's classic *Methods of Mathematical Physics vol. II.* Among those, the books of F. John [J] and Garabedian [G] are favorites of mine. In the last twenty years that mold has been broken several times. The courses of Bers, John, and Schecter [BJS], Agmon [A] and Lions [L] were trail blazers. They introduced recent methods to the next generation. None of these books was intended to be a broad introduction to partial differential equations, yet they set the stage for the next generation of texts, for example those by Treves [Tr], Chazarain and Piriou [CP], and Rauch [R]. In addition there are the encyclopedic treatises of Hörmander [H] and M. Taylor [Ta] and more specialized texts, notably those of Gilbarg and Trudinger [GT] and of Smoller [S]. The landscape of partial differential equations offerings is now quite rich, and the book of Evans is one of the very best. Among the introductory graduate texts, it is unique in giving a good perspective on strongly nonlinear phenomena.

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