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Symmetries and conservation laws for differential equations of mathematical physics, by I. S. Krasil'shchik and A. M. Vinogradov, American Mathematical Society, 1998, xiv + 333 pp., $29.00, ISBN 0-8218-0958-X

A century has now passed since the death of Sophus Lie. And yet, in many respects, his legacy has never been more alive. As I was writing these lines, confirmation appeared in the most recent issue (January 2000) of the *Bulletin*, surveying mathematical developments of the last century, where two of the first three book reviews are devoted to Lie's work. Despite universal acclaim for Lie's theory of continuous or Lie groups, many mathematicians remain only vaguely aware of Lie's original, overriding motivation — the study of differential equations. (An excellent source for the fascinating historical details is T. Hawkins, "Jacobi and the birth of Lie's theory of groups", *Arch. Hist. Exact Sci.* **42** (1991) 187–278.) Nevertheless, the remarkable range of applications of Lie groups to differential equations in geometry, in analysis, in physics, and in engineering over the past 40 years has resurrected Lie's original vision into one of the most active and rewarding fields of contemporary research.

The key results and ideas, many of which can be traced back to Lie's original work, include explicit, algorithmic determination of the symmetry group of any system of differential equations, integration of systems of ordinary differential equations by quadratures, determination of explicit solutions to nonlinear partial differential equations using symmetry reduction techniques, determination of conservation laws using Noether's Theorem, classification of differential invariants, invariant differential equations and variational problems, classification of integrable systems, Hamiltonian structures for both ordinary differential equations and evolution equations, linearization of nonlinear differential equations, equivalence of submanifolds and differential equations, contact transformations, boundary value problems, bifurcation theory, symmetry-preserving numerical methods and difference equations, etc., etc. Specific physical applications include fluid mechanics, magnetohydrodynamics, elasticity, general relativity, shock waves, solitons, image processing, control theory, and any aspect of physics and engineering where nonlinear differential equations play a significant role. The great advantage of the symmetry approach is that it is entirely algorithmic and can be implemented in most symbolic manipulation computer algebra systems.

The book under review joins a by now fairly extensive collection of recent texts detailing Lie's along with more contemporary contributions to the geometric study of differential equations. As is typical in a multi-authored text, the level of exposition and requirements imposed on the reader vary considerably from chapter to chapter. The first chapter gives a very readable and well-illustrated introduction to symmetry groups along with the underlying geometry of ordinary differential equations, culminating with Lie's symmetry-based explanation of explicit solution methods by quadratures. While much more modest in its aims than Arnol'd's *Geometrical Methods in the Theory of Ordinary Differential Equations* (Springer–Verlag, 1983), the introductory material is well-written and serves as a very good

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lead-in to the general theory. Chapter 2 introduces the basic geometrical structures on jet spaces, including the contact forms (or, as the book insists on slightly mislabeling them, "Cartan forms", which invites confusion with the important Cartan form in the variational calculus), contact transformations, symmetries and first integrals, along with applications to first order systems of partial differential equations, including some classical, but now slightly obscure, integration techniques.

Chapter 3 covers much the same ground, but now for higher order differential equations. The symmetries remain "classical" in the sense that they are prolongations of either geometrical transformations on the base manifold or, at most, first order contact transformations on the first jet bundle. (Bäcklund's Theorem restricts us to these cases.) The theory is amply illustrated by some explicit examples arising in nonlinear waves (Burgers' equation), acoustics (the Khokhlov–Zabolotskaya equation), and magnetohydrodynamics (the Kadmotsev-Pogutse equations). Although the applications and algorithmic computations are written in a form that can be understood by even the casual user, a more sophisticated level of abstraction is already evident in the theory, particularly in the chapter's final section on intrinsic versus extrinsic symmetries. Unfortunately, it is also at this stage that the explicit examples and applications become progressively more infrequent. The authors' unnecessarily confusing notation for derivatives, using, for example, p_{01} instead of the more palatable u_x, exacerbates the loss in clarity, since it is not so easy for the reader to decipher the differential equations being studied.

Chapter 4 deals with the theory and applications of higher or generalized symmetries (also sometimes called Lie–Bäcklund transformations, although this is a historical misnomer — they initially appear in Noether's famous paper on symmetries of variational problems). By now, the theory relies on a rather sophisticated and abstract machinery, leaving the applied reader trying to cope with complicated commutative diagrams and the use of obscure Russian letters in formulae. This is unfortunate, since, when written in a more down-to-earth manner, many of these concepts can be understood by the applications-oriented user of the theory and form some of the most fascinating aspects of the modern developments in the subject. I should also remark that the results in section 4 on evolution equations with higher order symmetries have been almost entirely superseded by the recent complete classification results of Beukers, Sanders and Wang, *J. Diff. Eq.* **147** (1998) 410–434, which, remarkably, rely on number-theoretic methods from Diophantine approximation theory!

A particular strength of the monograph is its presentation of the topological machinery of the variational bicomplex. The subject can trace its origins to the investigations in the nineteenth century of Helmholtz, who asked when a system of differential equations is the Euler-Lagrange system of equations for some variational problem. Further results by Mayer and Hirsch preceded the incisive investigations of Jesse Douglas (1941). Recent work by the authors of this text and others, particularly Anderson, Duchamp, Fels, and Thompson, show that even this special problem continues to be on the forefront of contemporary research. Early intimations that the inverse problem was part of a much larger machine can be found in the work of Dedecker, but it was the pathbreaking papers of Vinogradov, Tulczyjew, and, later, Tsujishita that laid the groundwork for the application of powerful topological tools to the geometric theory of partial differential equations and symmetries. Unfortunately, despite the important constructions and remarkable fundamental results, these early works were marred by an extremely cumbersome notation coupled with obscure and overly technical explanations.

The basic construction of the variational bicomplex is not so hard — it relies on the natural splitting of the space of differential forms depending on independent and dependent variables and their derivatives (the jet bundle coordinates) into horizontal and contact components. This endows the usual deRham complex with the structure of a bicomplex, and so the powerful homological algebra machinery, particularly spectral sequences, can be unleashed to compute topological quantities of interest, including conservation laws and variational structures. Chapter 5 reviews the basics of the variational bicomplex. While the theoretical discussion tends to get bogged down in technical, abstract details, there are quite a few noteworthy examples of computations, along with applications to the Hamiltonian structure of evolution equations, conservation laws, recursion operators, and the theorem of Magri on the complete integrability of bi-Hamiltonian systems (which, strangely, is stated without attribution). A significant weakness is that the monograph contains no introductory material on spectral sequences nor any elementary examples that would help the reader comprehend the complicated constructions outlined in this part.

The final chapter covers recent work on nonlocal symmetries and symmetries of integro-differential equations, based on the authors' notion of "covering". In essence, the construction requires introducing additional variables that convert the nonlocal terms into local expressions, to which the classical Lie theory can be applied. The one explicit application is to a system of integro-differential equations modeling coagulation. Given the amount of technical machinery required to understand the computations, the final result (translation and scaling symmetries, which could be easily guessed) is disappointing.

In summary, while I found some parts of this book quite enjoyable reading and would recommend them to students or colleagues, other parts would be rather tough going and of interest only to true aficionados. The authors' stated goals of making their earlier work accessible to a wide audience, including applied mathematicians, have, by and large, not been fully realized. Thus, we are still awaiting a *bona fide* comprehensive but readable introduction to the variational bicomplex and its many ramifications and applications.

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