BOOK REVIEWS

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Riemannian Geometry and Geometric Analysis, by J. Jost, Universitext, Springer, New York, 1995, 400 pp., $54.00, ISBN 3-540-57113-2

Riemannian Geometry, by P. Petersen, Graduate Texts in Mathematics, vol. 171, Springer-Verlag, New York, 1998, xvi + 432 pp., $49.95, ISBN 0-387-98212-4

Riemannian Geometry, by T. Sakai, Translations of Math. Monographs, vol. 149, Amer. Math. Soc., Providence, RI, 1995, xiv+358 pp., $59.00, ISBN 0-8218-0284-4

The three books under consideration treat advanced topics in differential, mostly Riemannian geometry. Each of them covers a vast number of topics, and it would be too ambitious to enumerate all of them here. Below I will, however, include a short description of the contents and structure of each of the books.

An excellent introduction to this review and the subject of Riemannian geometry is J. Kazdan's review [Kaz] of Chavel's book [Ch2], all the more so since this book is among the serious competitors of the three books considered here.

Each of the three books we consider here starts at an elementary level, but it is probably advisable to get some first exposure to Riemannian geometry from some other source. This remark applies the least to the book by Petersen, where the author spends some time discussing basic concepts in elementary examples. I think that all three books will be useful as reference books and as sources of inspiration for advanced students and their teachers. I now give a short description of the contents and structure of each of the books (in alphabetical order of their authors).

Short description of the contents of Jost's book

The book by Jost starts out with a chapter "Foundational Material", beginning with the definition of manifold on page 1 and ending, on pages 55–75, with a discussion of spin structures, spinor and Clifford bundles. This first chapter also contains a section on Riemannian metrics (pages 12–32). In this section Jost introduces and discusses the Riemannian distance, obtains the differential equation for geodesics as the Euler–Lagrange equation for the energy functional and discusses, among others, the exponential map and normal coordinates, the existence of geodesics in homotopy classes and the theorem of Hopf–Rinow. This list of topics maybe underlines the point I made above that the presentation is very dense.

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It seems to me that the structure of Jost's book is more determined by the logical structure of the material than by pedagogical or historical considerations. I don't want this remark to be understood as criticism. There is a lot which speaks in favour of building up a text that way. This structural point of view is very clear from the second chapter, where Jost discusses the Laplacian on forms and the representation of cohomology classes by harmonic forms. The theory of connections and curvature is not needed for this and therefore comes later. Elliptic regularity theory cannot be avoided though, and Jost reviews this theory in an appendix.

In the third chapter Jost starts with connections on vector bundles and their parallel transport and curvature. He then discusses some elementary facts about Yang–Mills functional, Chern–Weil theory, Chern classes and the Chern–Simons functional. He proceeds with the Levi–Civita connection and the different curvatures of Riemannian manifolds and proves Schur's characterizations of constant curvature and Einstein manifolds. After that Jost discusses connections and Dirac operators on Clifford and spinor bundles and explains and applies the Bochner method. In the last section of this chapter, Jost derives the Gauss equation for submanifolds and discusses minimal and totally geodesic submanifolds.

In the fourth chapter Jost derives the first and second variation of arc length and energy. As first applications he gets statements about the uniqueness of geodesics in negatively curved manifolds and Synge's theorem. He then introduces Jacobi fields, discusses conjugate points, and proves the Morse index and Bonnet–Myers theorem. The chapter ends with the Rauch comparison theorems and an application to (approximate) representations of functions in geodesic balls.

The first four chapters treat what Jost calls "the more elementary and basic aspects of the subject." Before the fifth chapter Jost places an interlude containing a discussion, without proofs, of some of the important results on the relation between geometry and topology, mainly applications of comparison techniques as in the fourth chapter, which are not covered in his book—the sphere theorem, to mention one.

In the fifth chapter Jost discusses critical point theory, including some critical point theory on path spaces. He introduces deRham cohomology and derives the Morse inequalities. As the main application he considers the energy functional on spaces of curves and proves the theorem of Fet and Lusternik on the existence of a closed geodesic on a compact Riemannian manifold. The discussion of the Sobolev space $\Lambda_0 = \Lambda_0(M)$ of closed H^1–curves on a Riemannian manifold M seems a bit of an overkill here, since in the arguments Jost uses finite dimensional approximations (as in Milnor's book).

The sixth chapter starts with a discussion of complex projective space and of some Kähler geometry. Then Jost introduces symmetric spaces and discusses some of their properties. Besides complex projective space the main example in the book is the homogeneous space $Sl(n,\mathbb{R})/SO(n)$.

As preparation for the last and probably main chapter of the book, Jost discusses, in Chapter 7, the Palais–Smale condition and verifies it for the energy functional on the space $\Lambda_0(M)$ in the case when M is compact. The application he derives from this could also be derived using finite dimensional approximations, and the Palais–Smale condition for the latter is obvious. However, the arguments given in the book may also be considered as model arguments with respect to harmonic maps.

The heart of the book is in the eighth chapter: harmonic maps between Riemannian manifolds. There is a section with results on the existence of harmonic maps with domain a surface (with or without boundary), and another section on the existence of harmonic maps when the range has nonpositive sectional curvature. Jost derives a Bochner formula for harmonic maps and discusses some of its applications to manifolds of nonpositive curvature.

Finally, in the ninth chapter Jost introduces the Ginzburg–Landau and the Seiberg–Witten functional and discusses some of their elementary aspects. There are appendices on elliptic partial differential equations and on covering spaces and fundamental group.

To many sections Jost has the appended "Perspectives" which contain historical remarks, outlines of developments and references for further reading related to the subjects of the section. Each chapter ends with a section containing exercises.

Short description of the contents of Petersen's book

Petersen assumes that the reader knows the elementary definitions of differential manifolds and starts, on line 1 of the text, with the definition of a Riemannian manifold. I think this is reasonable.

Petersen emphasizes the choice of the right coordinates for a given problem. Following this guideline, the first chapter contains a discussion of various local representations of Riemannian metrics. The main examples are warped products and doubly warped products. This latter class of metrics is rather broad. For example, writing the sphere as a doubly warped product in the right way, Petersen shows that the Hopf fibration to the complex projective space is a Riemannian submersion.

In the second chapter Petersen discusses the Levi–Civita connection and the different curvatures of Riemannian manifolds. He proves Schur's characterizations of constant curvature and Einstein manifolds. He also discusses distance functions and the relevant fundamental equations — Riccati, Gauss and Codazzi equation — for the corresponding level sets. Another point of view, related to the one about coordinates mentioned above, comes into effect here: throughout the book Petersen avoids variational techniques, Jacobi fields are not considered, and (hence) he introduces conjugate and focal points as those points, where the metric develops a singularity in coordinates adapted to the distance function. The second chapter ends with a short outline of the general theory of tensors. In the third chapter Petersen discusses various examples, mostly following the strategy of choosing good coordinates: he computes the curvature tensor of the sphere of radius r by using r as a distance function in the ambient Euclidean space; introduces the Schwarzschild metric, different models for hyperbolic space, and the Berger spheres; and computes the curvature tensor of complex projective space.

In the fourth chapter Petersen discusses hypersurfaces in Euclidean space. He proves Hadamard's theorem and discusses the existence of hypersurfaces with a given first and second fundamental form. There is also a discussion of the Chern–Gauss–Bonnet formula, and in the two dimensional case he presents a proof using the index of vector fields.

In the fifth chapter Petersen introduces the covariant derivative along curves and uses it to define parallel translation. Geodesics occur here for the first time, and following his point of view Petersen introduces them as autoparallel curves.

In Chapter 5 Petersen also discusses the Riemannian distance, exponential map, Gauss lemma, completeness (that is, the Hopf–Rinow theorem) and the injectivity radius.

In Chapter 6 Petersen starts with the comparison theory for distance functions. For manifolds of nonpositive sectional curvature he proves the Hadamard–Cartan theorem, Cartan's fixed point theorem and Preissmann's theorem. For manifolds of positive sectional curvature he shows the Bonnet–Myers theorem (case of sectional curvature), Synge's theorem and Klingenberg's estimate for the injectivity radius in even dimensions. These are the most important direct applications of the basic comparison techniques.

Before proceeding with other results from standard comparison geometry, there are now two chapters with a different flavour. Chapter 7 contains a discussion of Killing fields, Hodge Theory (a short outline) and Bochner's argument. This is applied in various cases: estimate of the first Betti number of compact manifolds with nonnegative Ricci curvature (Bochner's original application), rational cohomology of manifolds with nonnegative curvature operator (Gallot-Meyer) and others. In Chapter 8 Petersen discusses symmetric spaces. The main examples here are the real Grassmannnians, their duals and complex projective space.

In the ninth chapter Petersen discusses manifolds with lower bounds on the Ricci curvature. The basic result is the Bishop–Gromov volume comparison theorem. Besides the Bonnet–Myers theorem (for Ricci curvature) he proves Cheng's maximal diameter theorem, Gromov's estimate of the first Betti number and various extensions of the latter result by Anderson. Using, among others, Busemann functions and a general maximum principle due to Calabi, Petersen proves the splitting theorem of Cheeger and Gromoll.

In my opinion Chapter 10 is one of the central chapters of the book. Here Petersen discusses spaces of Riemannian manifolds and metrics, Gromov–Hausdorff distance, harmonic coordinates and convergence questions. He proves a general theorem about the compactness of certain Hölder classes of metrics, and Cheeger's finiteness theorem is an application. Petersen also discusses some related results of Anderson (in which Ricci curvature is considered).

In the last chapter, Chapter 11, Petersen returns to comparison results for sectional curvature. He discusses critical points of distance functions and proves the sphere theorems of Rauch–Berger–Klingenberg and Grove–Shiohama. He also proves the soul theorem of Cheeger and Gromoll, Gromov's Betti number theorem and the homotopy finiteness theorem of Grove and Petersen.

There are three appendices: one on deRham cohomology, one on principal bundles and one on spinor bundles. Each chapter ends with a section "Further Study" containing references in which one can find more on the material covered in the chapter and a section with exercises.

Short description of the contents of Sakai's book

In Chapter 1 Sakai introduces the language of manifolds and vector bundles and defines connections on vector bundles. In Chapter 2 he starts by defining Riemannian metrics and discusses the associated Riemannian distance, Levi–Civita connection and Christoffel symbols. He defines geodesics as autoparallel curves and discusses the exponential map. Next he introduces curvature tensors and the Jacobi equation and derives the Gauss lemma and the first variation of arc length, leading to the variational characterization of geodesics. Further results from this chapter are the determination of the differential of the exponential map, Schur's characterization of constant curvature and Einstein manifolds and Gauss and Codazzi equations for submanifolds. In a section entitled "From the point of view of the tangent bundle" Sakai discusses the geodesic flow on the tangent bundle. Finally, there are sections on the divergence formula and on Riemannian submersions with a discussion of complex projective space.

In Chapter 3 Sakai starts with completeness (Hopf–Rinow theorem), derives first and second variation of the energy functional, proves the Morse index theorem and derives estimates for the index of geodesics. As an application he obtains Myer's diameter theorem (case of sectional curvature). He then proceeds with the cut locus and injectivity radius and shows Warner's theorem that for a complete and simply connected Riemannian manifold the first conjugate locus and cut locus coincide if the order of the first conjugate locus is always ≥ 2. Finally, he proves Ambrose's theorem on parallel translation of Riemannian curvature and discusses Killing fields and isometries, holonomy and the deRham decomposition theorem.

Chapter 4 mainly serves as preparation for the discussion of relations between geometry and topology of the following chapter. After a short discussion of constant curvature spaces, Sakai derives the technical results of comparison geometry: Rauch comparison theorem for Jacobi fields and the corresponding results for the Hessian of distance functions, Bishop–Gromov volume comparison, Heintze–Karcher volume comparison and Toponogov's triangle comparison theorem. As first applications Sakai discusses the Bonnet–Myers diameter theorem (case of Ricci curvature) together with Cheng's maximal diameter theorem and Cheeger's estimate on the length of closed geodesics. In the last two sections of this chapter, Sakai discusses convexity and symmetric spaces respectively.

The central chapter of the book is Chapter 5, devoted to classical results of global Riemannian geometry. Sakai starts with a discussion of the fundamental group: finiteness of the fundamental group as a consequence of (the assumptions of) the Bonnet–Myers diameter theorem and the results of Milnor and Schwarz on the growth of the fundamental group of compact Riemannian manifolds of negative sectional curvature respectively nonnegative Ricci curvature. Sakai also discusses Klingenberg's injectivity radius estimate in even dimensions and Bochner's result on the first Betti number. The second section is devoted to the Rauch–Berger–Klingenberg sphere theorem, the Grove–Shiohama diameter sphere theorem and Berger's rigidity theorem. In the following section Sakai proves the soul and splitting theorems of Cheeger and Gromoll, and in the final section the Hadamard–Cartan theorem, Cartan's fixed point theorem and the flat torus theorem of Gromoll–Wolf and Lawson–Yau.

In the final chapter, Chapter 6, Sakai discusses isoperimetric inequalities and spectral theory of the Laplace operator. In the first section he shows estimates of the Cheeger isoperimetric constant by Gallot and Bérard–Besson–Gallot respectively, in the second section the isoembolic inequality of Berger and Kazdan. After that he starts with the general theory of the spectrum of the Laplacian and determines the spectrum of round spheres and flat tori. He proves the Lichnerowicz–Obata theorem and Cheeger's and other estimates of the first non–zero eigenvalue of the Laplacian. Finally, he discusses the fundamental solution of the heat equation (with a sketch of the proof of its existence). The fundamental solution was used before in the general theory of the spectrum of the Laplacian.

There are appendices on, among others, curvature tensors, homogeneous spaces, Hodge theory and compactness and convergence theory in spaces of Riemannian manifolds. There are exercises throughout the text, at the end of each chapter another section with problems and a section with notes on the references.

In comparing the three books, one can say that their styles are certainly somewhat different, due in part to the different temperaments of the authors. All three books are rather densely written, and more details are left to the reader in the books of Jost and Petersen. Any of the three books is a good source for teaching a somewhat advanced class in differential geometry and certainly contains enough material for a one-year course. In such a case the books would also be very useful companions for the students. They are also good sources for the working differential geometer. In any case they are fine books and worthwhile additions to any differential geometer's library.

All of this being said, I would like to include — for the convenience of the reader and to put everything in perspective — a short and incomplete list of available books and texts in Riemannian geometry. The books [dC], [La] and [ON] are more elementary than the ones considered here, but they still contain some advanced topics. The books [Ch2], [GHL] and [Kl] are on a comparable level and discuss similar topics, except mainly for the discussion of harmonic maps in Jost's book and for much of the discussion of Ricci curvature and convergence and compactness questions in Petersen's book. The five volumes [Sp] by Spivak are of a different nature; they do not contain, for example, an in-depth discussion of comparison techniques. The book [Mi] of Milnor still contains one of the most readable introductions into the foundations of Riemannian geometry, and Milnor's presentation of the critical point theory of the energy functional on path spaces is not surpassed in the present books. Furthermore, I mention that the second edition of Gromov's book, one of the most influential texts in differential geometry of the last decade, just appeared in print [Gro].

There is also tough competition from another side. There are very good articles on specific, selected topics in Riemannian geometry. The articles [Kar] and [Me] by Karcher and Meyer respectively contain a very good account of many results from global Riemannian geometry, starting from proofs of the Rauch and Toponogov comparison theorems. There are also very good other lecture notes, for example by Eschenburg [Es], Grove [Gr], Jost [Jo2] and Li [Li]. In the references I have included some books which are available and devoted to special topics related to topics in the books by Jost, Petersen and Sakai. To these belong [Be], [BZ], [Ch1], [Eb] and [Wu].

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