# BOOK REVIEWS

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*Simplicial homotopy theory*, by Paul G. Goerss and John F. Jardine, Progress in Mathematics, vol. 174, Birkhäuser-Verlag, Basel, 1999, xv + 510 pp., $69.95, ISBN 3-7643-6064-X

Simplicial sets, originally called complete semi-simplicial complexes or c.s.s. complexes, first appeared in a paper of Eilenberg and Zilber [EZ] in 1950. The resulting category generalized simplicial complexes, the singular complex of a space (used for calculating singular homology and cohomology), and the complexes used in defining the cohomology of groups. The idea was quite simple. A semi-simplicial complex consists of a set $K_n$ for each $n \geq 0$ whose elements are called "$n$-simplices", and face maps

$$\partial_i : K_n \to K_{n-1} \qquad 0 \leq i \leq n$$

where $\partial_i(x)$ is the $i^{\text{th}}$ "face" of the "simplex" $x$. These satisfy the relations that occur in a simplicial complex ($\partial_i \partial_j = \partial_{j-1}\partial_i$ if $i < j$). A complete semi-simplicial complex also has degeneracies:

$$s_i : K_n \to K_{n+1} \qquad 0 \leq i \leq n$$

corresponding to the various order preserving simplicial maps from the $(n+1)$-simplex onto the $n$-simplex. There is a complex set of relations between the $s_i$ and $\partial_i$, but the entire setup has an elegant description in the language of category theory. Let $\Delta$ be the category with one object $[n] = \{0, 1, \ldots, n\}$ for each $n \geq 0$ where $\hom([m], [n])$ is the set of functions $\alpha : [m] \to [n]$ such that $\alpha(a) \geq \alpha(b)$ if $a \geq b$. Then a c.s.s. complex or simplicial set is just a contravariant functor from $\Delta$ to the category of sets. If $K$ is such a functor, $K_n = K\big([n]\big)$ and $\partial_i$ is obtained by applying $K$ to the 1-1 function $[n-1] \to [n]$ which misses $i$. Similarly $s_i : K_n \to K_{n+1}$ is obtained by applying $K$ to the onto function $\alpha : [n+1] \to [n]$ with $\alpha(i) = \alpha(i+1)$. We will write $\mathcal{S}$ for the category of simplicial sets and simplicial maps. It is clear that we can consider contravariant functors from $\Delta$ to other categories, and we often do, providing us with simplicial groups, simplicial spaces, and even simplicial simplicial sets (or bisimplicial sets).

The singular complex $S(X)$ of a space $X$ is given by

$$S(X)_n = \{f : \Delta^n \longrightarrow X | f \text{ is continuous}\}$$

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where $\Delta^n$ is the standard $n$ simplex. An order preserving map $\alpha : [m] \to [n]$ clearly defines a linear map $\Delta^m \to \Delta^n$ which by composition defines a function

$$S(X)_n \longrightarrow S(X)_m.$$

Clearly $X \to S(X)$ is a covariant functor from the category of $\mathcal{T}$ topological spaces to $\mathcal{S}$.

In 1956, Kan [K1] introduced the extension condition: If $n$ elements of $K_{n-1}$ fit together like all but one of the faces of an $n$ simplex, then there is an $n$-simplex in $K_n$ with these faces. The singular complex $S(X)$ satisfies this condition since the union of all but one face of $\Delta^n$ is a retract of $\Delta^n$. The significance of this condition is that, in its presence, one can introduce the notion of homotopy - both among certain simplices in $K$ and between simplicial maps. Complexes satisfying the extension condition are called Kan complexes.

In 1957, Milnor [M1] constructed a functor left adjoint to $S$. This functor, called the realization, assigned to each $K \in \mathcal{S}$ a CW complex $|K|$. The adjointness condition:

$$\operatorname{map}_{\mathcal{S}}\big(K, S(X)\big) \cong \operatorname{map}_{\mathcal{T}}\big(|K|, X\big)$$

determines maps $\epsilon : K \longrightarrow S\big(|K|\big)$ and $\eta : \big|S(X)\big| \longrightarrow X$. In case $K$ is a Kan complex, $\epsilon$ is a homotopy equivalence, and in case $X$ is the homotopy type of a CW complex, $\eta$ is a homotopy equivalence. Since both functors preserve homotopy and $S(X)$ is a Kan complex, this establishes an equivalence between the homotopy category of CW complexes and the homotopy category of Kan complexes - replacing a topological category by a combinatorial one.

Also in 1957, Brown [B] used simplicial techniques to prove that the homotopy groups of a finite simply connected CW complex are each finitely computable. The study of simplicial sets was an active subject throughout the 1950's with papers by Barratt, Gugenheim, Kan and Moore. There was some hope that the combinatorial nature of the simplicial category might lead to an algorithm for calculating homotopy groups. Although there seems to be some renewed interest in this vein, no one has yet been able to perform the simplest of calculations directly from these tools. This was the state of the subject in 1960. It seemed like a dead-end until the appearance in 1963 of work of Curtis [C1] on the lower central series of a simplicial group leading to the unstable Adams spectral sequence in 1966 [6A]. These results have had a powerful impact on algebraic topology.

The first exposition on simplicial theory was *Simplicial Objects in Algebraic Topology* by May [Ma, 1967]. Finally students had a mature organization of the subject. It did not, however, contain any of the developments since 1960. It also had a misfortune of timing. In that same year Quillen published "Homotopical Algebra", a paper which would have profound influence on simplicial theory. The equivalence of the homotopy category of Kan complexes and that of CW complexes insures that problems of homotopy classification can be solved in either category and the solution applies to both. Along the way, however, we often perform constructions in the topological (or simplicial) category before passing to homotopy classes: constructions such as limits, co-limits, taking the homotopy fiber or cofiber of a map, or forming Toda brackets. It is important to know that these constructions correspond under the equivalence. Quillen defined a model category to be a category with finite limits and colimits and certain classes of maps called fibrations, cofibrations, and weak equivalence subject to axioms expressing the usual relationship between these concepts. From these notions one is able to define homotopy and pass to the homotopy category. It turns out that weak equivalences become equivalences in the homotopy category. Given two model categories, there is a notion called Quillen equivalence. The definition is quite technical: in essence there is a pair of adjoint functors which preserve the structure and in particular equivalences such as $\epsilon$ and $\eta$ above.

This book is welcome and fills a sorely needed gap. No text or expository monograph has been published on this material in over 30 years. Much has happened during these years. This book of 510 pages is efficient. It is quite compact and everywhere dense.

Chapters 1 and 2 form a basis for the book. Chapter 1 covers most of the classical material including Kan complexes, minimal fibrations and ends by establishing that the category of simplicial sets is a closed model category. Chapter 2 discusses model categories. In particular, a version of the Whitehead theorem is established in an arbitrary closed model category (a weak equivalence between objects that are both fibrant and cofibrant is a homotopy equivalence). In the case of simplicial sets, the set of simplicial maps $\text{hom}(X, Y)$ is the set of vertices of a simplicial set (the function complex $\text{Hom}(X, Y)$). A simplicial model category is defined to be one in which a function complex can be defined in an appropriate way, and criteria for the existence of this structure are established. The final result here is Quillen's derived functor theorem.

Chapter 3 is quite miscellaneous. The fundamental groupoid of a simplicial set, as a functor, is seen to be adjoint to the classifying space construction on small categories. The Hurewicz theorem is proven (using the Serre spectral sequence which is derived in the next chapter). The Dold-Kan correspondence is established between simplicial Abelian groups and chain complexes. Chapter 4 is on bisimplicial sets. Three different closed model structures are provided, all with the same homotopy theory. The discussion moves to the Serre spectral sequence, group completion, and Quillen's Theorem B. A generalized Eilenberg-Zilber theorem is proven as well as the Bousfield-Friedlander Theorem. Chapter 5 discusses simplicial groups and group actions. In particular, principal fibrations and classifying spaces are obtained as well as Milnor's FK construction. Chapter 6 discusses towers and inverse limits; a model structure is provided and there is a careful analysis of Postnikov towers. Chapter 7 is on "Reedy theory" - a Reedy structure is a closed model structure on the category of simplicial objects in a suitable closed model category. This first appeared in Chapter 4. It also applies (by duality) to cosimplicial spaces (cosimplicial simplicial sets) in which case it is the Bousfield-Kan structure. Chapter 8 covers the Bousfield-Kan spectral sequence for the homotopy of a cosimplicial space. Chapter 9 discusses realization of homotopy coherent diagrams and the Dwyer-Kan theorem. Chapter 10 is on localization, either with respect to a homology theory or a map.

This book will be a useful source for the experts: those with experience with simplicial techniques and model categories. A graduate student might find it rough going - one should certainly be thoroughly familiar with Mac Lane's *Categories for the Working Mathematician* or keep it handy as it seems to be the (only) prerequisite.

A student with no algebraic topology background could conceivably read this book, although I would not recommend it. In fact, if one is not at all familiar with simplicial theory, May's book [Ma] or Curtis [C2] would be a gentler introduction to the subject.

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Brayton Gray
University of Illinois at Chicago
*E-mail address*: brayton@uic.edu